Non-unitary deviation from the tri-bimaximal lepton mixing and its implications on neutrino oscillations

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Abstract

We propose a new pattern of the neutrino mixing matrix which can be parametrized as the product of an arbitrary Hermitian matrix and the well-known tri-bimaximal mixing matrix. In this scenario, nontrivial values of the smallest neutrino mixing angle θ_{13} and the CP-violating phases entirely arise from the non-unitary corrections. We present a complete set of series expansion formulas for neutrino oscillation probabilities both in vacuum and in matter of constant density. We do a numerical analysis to show the non-unitary effects on neutrino oscillations. The possibility of determining small non-unitary perturbations and CP-violating phases is discussed by measuring neutrino oscillation probabilities and constructing "deformed unitarity triangles". Some brief comments on the non-unitary neutrino mixing matrix in the type-II seesaw models are also given.

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I. MOTIVATION

Recent solar [1], atmospheric [2], reactor [3] and accelerator [4] neutrino experiments have convincingly verified the hypothesis of neutrino oscillations, which can naturally happen if neutrinos are slightly massive and lepton flavors are mixed. This discovery indicates the existence of new physics beyond the Standard Model (SM). Although it is easy to add either Dirac or Majorana neutrino mass terms to the SM, it is highly non-trivial to reveal the essential meaning behind such terms and find a natural and qualitative explanation of the smallness of neutrino masses. A complete theory of neutrino mass generation has been lacking and is eagerly desirable.

A low-energy effective theory responsible for the generation of neutrino masses might give rise to slight violation of the unitarity of the neutrino mixing matrix. If three light neutrinos are mixed with other degrees of freedom (e.g., with the light sterile neutrinos [5], the heavy Majorana neutrinos [6], or the whole tower of Kaluza-Klein states in some models with extra dimensions [7]), their 3×3 flavor mixing matrix V appearing in the SM charged-current interactions will in general be non-unitary. Therefore, the deviation of V from unitarity can serve as an indicator of new physics beyond the SM.

A generic non-unitary neutrino mixing matrix V can be parametrized as $V = H \cdot V_0$ [1], where H is a Hermitian matrix which can be written as

$$H = \begin{pmatrix} a & \hat{\kappa}_{12} & \hat{\kappa}_{13} \\ \hat{\kappa}^*_{12} & b & \hat{\kappa}_{23} \\ \hat{\kappa}^*_{13} & \hat{\kappa}^*_{23} & c \end{pmatrix} , \tag{1}$$

with a, b, c being real and $\hat{\kappa}_{ij}$ $(ij = 12, 13, 23)$ being complex, and V_0 is a unitary matrix which is usually parametrized in terms of three mixing angles and one Dirac CP-violating phase [8] as

$$V_0 = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix} . \tag{2}$$

[1] There is no unique way to parametrize a generic 3×3 flavor mixing matrix. For example, the non-unitary matrix V can also be expressed as $V = A \cdot V_0^{'}$, where A is a lower (or upper) triangle matrix and $V_0^{'}$ is a unitary matrix [9]. Although different parametrizations are mathematically equivalent, they may have their own advantages in describing different phenomena of neutrino physics.

Here we have omitted two Majorana CP-violating phases of V_0 since they are irrelevant to neutrino oscillations.

Given the lepton mixing matrix V, the constraints on the moduli of the elements of $VV^\dagger$ have been deduced in Ref. [10] by combining the experimental data on both neutrino oscillations and precision electroweak tests:

$$|VV^\dagger| \approx \begin{pmatrix} 0.994 \pm 0.005 & < 7.0 \times 10^{-5} & < 1.6 \times 10^{-2} \\ < 7.0 \times 10^{-5} & 0.995 \pm 0.005 & < 1.02 \times 10^{-2} \\ < 1.6 \times 10^{-2} & < 1.02 \times 10^{-2} & 0.995 \pm 0.005 \end{pmatrix} . \tag{3}$$

It is then easy to find that $a, b, c \sim 1$, $|\hat{\kappa}_{12}| \lesssim 3.5 \times 10^{-5}$, $|\hat{\kappa}_{13}| \lesssim 8.0 \times 10^{-3}$, and $|\hat{\kappa}_{23}| \lesssim 5.1 \times 10^{-3}$ should hold. We can further denote a, b, c as $a = 1 + \epsilon_a$, $b = 1 + \epsilon_b$ and $c = 1 + \epsilon_c$; namely,

$$H \equiv \mathbf{1} + \epsilon = \mathbf{1} + \begin{pmatrix} \epsilon_a & \hat{\kappa}_{12} & \hat{\kappa}_{13} \\ \hat{\kappa}^*_{12} & \epsilon_b & \hat{\kappa}_{23} \\ \hat{\kappa}^*_{13} & \hat{\kappa}^*_{23} & \epsilon_c \end{pmatrix} , \tag{4}$$

where ϵ_a, ϵ_b, ϵ_c are all real. Bounds on ϵ_a, ϵ_b and ϵ_c can also be obtained from Eq. (3): $|\epsilon_a| \lesssim 5.5 \times 10^{-3}$, $|\epsilon_b| \lesssim 5.0 \times 10^{-3}$ and $|\epsilon_c| \lesssim 5.0 \times 10^{-3}$.

The effects of non-unitarity of V on neutrino oscillations have been discussed in some literature [9, 10, 11, 12]. In particular, the authors of Ref. [12] have used the same parametrization of V as given above and explored the effects of CP violation induced by those non-unitary complex parameters in neutrino oscillations.

In this paper, we start from an intriguing point of view that the realistic neutrino mixing matrix V might result from a non-unitary correction to the well-known tri-bimaximal mixing pattern [13]. The latter is compatible with current experimental data very well and can be derived from a number of flavor symmetries and their spontaneous or explicit breaking mechanisms [14]. Instead of building a specific neutrino model to realize such a phenomenological conjecture, here we shall concentrate on the consequences of V on neutrino oscillations.

In our new neutrino mixing scenario, $V = H \cdot V_0$ with V_0 being given by

$$V_0 = \begin{pmatrix} 2/\sqrt{6} & 1/\sqrt{3} & 0 \\ -1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \\ 1/\sqrt{6} & -1/\sqrt{3} & 1/\sqrt{2} \end{pmatrix} \tag{5}$$

and H being an arbitrary Hermitian matrix as shown in Eq. (1) or (4). To be specific, the non-unitary neutrino mixing matrix V reads

$$V = \begin{pmatrix} \frac{2}{\sqrt{6}}\,a - \frac{1}{\sqrt{6}}\,(\hat{\kappa}_{12} - \hat{\kappa}_{13}) & \frac{1}{\sqrt{3}}\,a + \frac{1}{\sqrt{3}}\,(\hat{\kappa}_{12} - \hat{\kappa}_{13}) & \frac{1}{\sqrt{2}}\,(\hat{\kappa}_{12} + \hat{\kappa}_{13}) \\ -\frac{1}{\sqrt{6}}\,b + \frac{1}{\sqrt{6}}\,(2\hat{\kappa}^*_{12} + \hat{\kappa}_{23}) & \frac{1}{\sqrt{3}}\,b + \frac{1}{\sqrt{3}}\,(\hat{\kappa}^*_{12} - \hat{\kappa}_{23}) & \frac{1}{\sqrt{2}}\,b + \frac{1}{\sqrt{2}}\hat{\kappa}_{23} \\ \frac{1}{\sqrt{6}}\,c + \frac{1}{\sqrt{6}}\,(2\hat{\kappa}^*_{13} - \hat{\kappa}^*_{23}) & -\frac{1}{\sqrt{3}}\,c + \frac{1}{\sqrt{3}}\,(\hat{\kappa}^*_{13} + \hat{\kappa}^*_{23}) & \frac{1}{\sqrt{2}}\,c + \frac{1}{\sqrt{2}}\hat{\kappa}^*_{23} \end{pmatrix} . \qquad (6)$$

It is clear that the parameters of H lead simultaneously to the unitarity violation and the deviation from V_0. The resulting smallest mixing angle θ_{13} to be measured in reactor $\bar{\nu}_e \to \bar{\nu}_e$ oscillation experiments is attributed to the small parameters $\hat{\kappa}_{12}$ and $\hat{\kappa}_{13}$. So are the CP-violating phases of V.

The remaining parts of this paper are organized as follows. In sections II and III, we develop a complete set of series expansion formulas for neutrino oscillation probabilities both in vacuum and in matter of constant density, respectively, by taking account of the non-unitary mixing matrix V given in Eq. (6). In section IV, we discuss the possibility of determining some parameters of H by constructing the "deformed unitarity triangles". Section V is devoted to a short discussion about incorporating our parametrization of the non-unitary neutrino mixing matrix into a generic type-II seesaw model. Finally some conclusions are drawn in section VI.

II. NEUTRINO OSCILLATIONS IN VACUUM

Suppose that the non-unitary V in Eq. (6) describes the mixing between the neutrino fields in the mass basis and those in the flavor basis,

$$\nu_\alpha = V_{\alpha i}\, \nu_i \, , \qquad (7)$$

where $\alpha = e, \mu, \tau$ and $i = 1, 2, 3$. The probability of neutrino oscillation $\nu_\alpha \to \nu_\beta$ ($P_{\alpha\beta}$) can be derived in a similar way to that in the unitary case. The procedures of deriving the formulas for neutrino oscillation probabilities can be found in Ref. [10]. Here we follow another way, in order to be concise. The derivation may be easily understood as follows.

A typical neutrino oscillation process $\nu_\alpha \to \nu_\beta$ can be divided into three parts [15]: 1) ν_α being produced at the source through the charged-current interaction which can be denoted as $W \to \bar{l}_\alpha \nu_\alpha$. Here, ν_α is a superposition of mass eigenstates ν_i; 2) ν_i propagates from the

source to the detector; 3) ν_β (a superposition of ν_i) being catched by the detector through the charged-current interaction $\nu_\beta W \to l_\beta$. Therefore, the amplitude of the neutrino oscillation $\nu_\alpha \to \nu_\beta$ can be correspondingly divided into three parts:

$$A(\nu_\alpha \to \nu_\beta) = \sum_i A(W \to \bar{l}_\alpha \nu_i) \mathrm{Prop}(\nu_i) A(\nu_i W \to l_\beta) \ . \tag{8}$$

In the case of non-unitary neutrino mixing, it follows that $A(W \to \bar{l}_\alpha \nu_i) = V^*_{\alpha i}/\sqrt{(VV^\dagger)_{\alpha\alpha}}$. The factor $1/\sqrt{(VV^\dagger)_{\alpha\alpha}}$ ensures that the total rate $P(W \to \bar{l}_\alpha \nu_\alpha) \equiv \sum_i |A(W \to \bar{l}_\alpha \nu_i)|^2 = 1$. Similarly, we have $A(\nu_i W \to l_\beta) = V_{\beta i}/\sqrt{(VV^\dagger)_{\beta\beta}}$. The expression of $\mathrm{Prop}(\nu_i)$ is the same as that in the unitary case: $\mathrm{Prop}(\nu_i) = \exp(-im_i^2 L/2E_\nu)$. Finally, the amplitude of the neutrino oscillation $\nu_\alpha \to \nu_\beta$ is given by

$$A(\nu_\alpha \to \nu_\beta) = \frac{1}{\sqrt{\left(VV^\dagger\right)_{\alpha\alpha} \left(VV^\dagger\right)_{\beta\beta}}} \sum_i V^*_{\alpha i} e^{-im_i^2 \frac{L}{2E_\nu}} V_{\beta i} \ . \tag{9}$$

Then $P_{\alpha\beta}$, the probability of neutrino oscillation $\nu_\alpha \to \nu_\beta$, is given by

$$\begin{aligned} P_{\alpha\beta} \ &= \ |A(\nu_\alpha \to \nu_\beta)|^2 \ = \ \frac{\left|\displaystyle\sum_i V^*_{\alpha i} e^{-im_i^2 \frac{L}{2E_\nu}} V_{\beta i}\right|^2}{\left(VV^\dagger\right)_{\alpha\alpha} \left(VV^\dagger\right)_{\beta\beta}} \\ &= \ \frac{1}{\left(VV^\dagger\right)_{\alpha\alpha} \left(VV^\dagger\right)_{\beta\beta}} \left[|\left(VV^\dagger\right)_{\alpha\beta}|^2 - 4\sum_{j<i} A^{ij}_{\alpha\beta} \sin^2\Delta_{ij} - 2\sum_{j<i} J^{ij}_{\alpha\beta} \sin 2\Delta_{ij} \right] \\ &= \ \frac{1}{\left(VV^\dagger\right)_{\alpha\alpha} \left(VV^\dagger\right)_{\beta\beta}} \cdot \Big[|\left(VV^\dagger\right)_{\alpha\beta}|^2 - 4A^{21}_{\alpha\beta}\sin^2\Delta_{21} - 4A^{31}_{\alpha\beta}\sin^2\Delta_{31} - 4A^{32}_{\alpha\beta}\sin^2\Delta_{32} \\ &\qquad -2J^{21}_{\alpha\beta}\sin 2\Delta_{21} - 2J^{31}_{\alpha\beta}\sin 2\Delta_{31} - 2J^{32}_{\alpha\beta}\sin 2\Delta_{32} \Big] \ , \end{aligned} \tag{10}$$

where $\Delta_{ij} \equiv \Delta m^2_{ij} L/(4E_\nu)$ with $\Delta m^2_{ij} \equiv m_i^2 - m_j^2$, and $A^{ij}_{\alpha\beta} = \mathrm{Re}[V_{\alpha i} V_{\beta j} V^*_{\alpha j} V^*_{\beta i}]$, $J^{ij}_{\alpha\beta} = \mathrm{Im}[V_{\alpha i} V_{\beta j} V^*_{\alpha j} V^*_{\beta i}]$. We can further absorb the renormalization factor $1/\left(VV^\dagger\right)_{\alpha\alpha}\left(VV^\dagger\right)_{\beta\beta}$ into the redefinitions of $A^{ij}_{\alpha\beta}$ and $J^{ij}_{\alpha\beta}$ and rewrite the above equation as

$$\begin{aligned} P_{\alpha\beta} \ &= \ \frac{|\left(VV^\dagger\right)_{\alpha\beta}|^2}{\left(VV^\dagger\right)_{\alpha\alpha} \left(VV^\dagger\right)_{\beta\beta}} - 4\hat{A}^{21}_{\alpha\beta}\sin^2\Delta_{21} - 4\hat{A}^{31}_{\alpha\beta}\sin^2\Delta_{31} - 4\hat{A}^{32}_{\alpha\beta}\sin^2\Delta_{32} \\ &\qquad -2\hat{J}^{21}_{\alpha\beta}\sin 2\Delta_{21} - 2\hat{J}^{31}_{\alpha\beta}\sin 2\Delta_{31} - 2\hat{J}^{32}_{\alpha\beta}\sin 2\Delta_{32} \ , \end{aligned} \tag{11}$$

where $\hat{A}^{ij}_{\alpha\beta} = A^{ij}_{\alpha\beta}/(VV^\dagger)_{\alpha\alpha}(VV^\dagger)_{\beta\beta}$, $\hat{J}^{ij}_{\alpha\beta} = J^{ij}_{\alpha\beta}/(VV^\dagger)_{\alpha\alpha}(VV^\dagger)_{\beta\beta}$.

The first term in Eq. (11) is the so-called "zero-distance" term. It means that at $L = 0$ we have

$$P_{\alpha\beta}(L=0) \ = \ \frac{|\left(VV^\dagger\right)_{\alpha\beta}|^2}{\left(VV^\dagger\right)_{\alpha\alpha} \left(VV^\dagger\right)_{\beta\beta}} \ . \tag{12}$$

Note that if $\alpha = \beta$, $P_{\alpha\beta}(L=0) = 1$; namely, there are no "zero-distance" effects in the disappearance experiments. If $\alpha \neq \beta$, the oscillation probability $P_{\alpha\beta}(L=0)$ is in general nonzero, that is the "zero-distance" effect. One can find that although it is nonzero, this term add only a tiny constant to the oscillation probability, and does not change the oscillatory behavior. In our scenario, we have $P_{e\mu}(L=0) \approx 4|\hat{\kappa}_{12}|^2$, $P_{e\tau}(L=0) \approx 4|\hat{\kappa}_{13}|^2$ and $P_{\mu\tau}(L=0) \approx 4|\hat{\kappa}_{23}|^2$. Another significant difference between the non-unitary and unitary cases is: if the mixing matrix V is non-unitary, there may exist 9 different Jarlskog invariants $J^{ij}_{\alpha\beta}$ corresponding to 3 different oscillation channels instead of a unique Jarlskog in the unitary case.

If $\Delta_{21} \ll 1$ is satisfied, we can expand Eq. (11) as

$$
\begin{aligned}
P_{\alpha\beta} \approx\; & \frac{|\left(VV^\dagger\right)_{\alpha\beta}|^2}{(VV^\dagger)_{\alpha\alpha}(VV^\dagger)_{\beta\beta}} \\
& -4\left(\hat{A}^{21}_{\alpha\beta} + \hat{A}^{32}_{\alpha\beta}\right)\Delta^2_{21} - 4\left(\hat{A}^{31}_{\alpha\beta} + \hat{A}^{32}_{\alpha\beta}\right)\sin^2\Delta_{31} \\
& +4\hat{A}^{32}_{\alpha\beta}\left(\Delta_{21}\sin 2\Delta_{31} + 2\Delta^2_{21}\sin^2\Delta_{31}\right) \\
& -4\left(\hat{J}^{21}_{\alpha\beta} - \hat{J}^{32}_{\alpha\beta}\right)\Delta_{21} - 2\left(\hat{J}^{31}_{\alpha\beta} + \hat{J}^{32}_{\alpha\beta}\right)\sin 2\Delta_{31} \\
& -4\hat{J}^{32}_{\alpha\beta}\left(\Delta^2_{21}\sin 2\Delta_{31} - 2\Delta_{21}\sin^2\Delta_{31}\right) \;,
\end{aligned}
\tag{13}
$$

In our calculations, we find that although all the nine $\hat{J}^{ij}_{\alpha\beta}$ are of $\mathcal{O}(\hat{\kappa}_{ij})$, only $\hat{J}^{32}_{\mu\tau} + \hat{J}^{31}_{\mu\tau}$ is of $\mathcal{O}(\hat{\kappa}_{ij})$ while $\hat{J}^{32}_{e\mu} + \hat{J}^{31}_{e\mu}$ and $\hat{J}^{32}_{e\tau} + \hat{J}^{31}_{e\tau}$ are both of $\mathcal{O}(\hat{\kappa}^2_{ij})$. This observation means that the most sensitive way at short baseline neutrino oscillation experiments to detect CP violation is to measure the $\nu_\mu \to \nu_\tau$ channel. Such a point was also pointed out in Refs. [9, 12].

Here we present a complete set of formulas for neutrino oscillation probabilities $P_{\alpha\beta}$ to the second order in powers of $\hat{\kappa}_{12}$, $\hat{\kappa}_{13}$, $\hat{\kappa}_{23}$, ϵ_a, ϵ_b, ϵ_c and Δ_{21}. These formulas are good approximations for the Δ_{31}-dominated oscillations, i.e., for neutrino oscillation experiments with relative short baselines and relatively high energies.

$$
P_{ee} \approx 1 - 2|\hat{\kappa}_{12} + \hat{\kappa}_{13}|^2\sin^2\Delta_{31} - \frac{8}{9}\Delta^2_{21} \;, \tag{14}
$$

$$
\begin{aligned}
P_{\mu\mu} \approx\; & 1 - \left[1 - 4\left(\mathrm{Re}[\hat{\kappa}_{23}]\right)^2\right]\sin^2\Delta_{31} + \frac{2}{3}\left(1 + 2\mathrm{Re}[\hat{\kappa}_{12}]\right)\Delta_{21}\sin 2\Delta_{31} \\
& -\frac{4}{9}\left(2 - 3\sin^2\Delta_{31}\right)\Delta^2_{21} \;,
\end{aligned}
\tag{15}
$$

$$
\begin{aligned}
P_{\tau\tau} \approx\; & 1 - \left(1 - 4\left(\mathrm{Re}[\hat{\kappa}_{23}]\right)^2\right)\sin^2\Delta_{31} + \frac{2}{3}\left(1 - 2\mathrm{Re}[\hat{\kappa}_{13}]\right)\Delta_{21}\sin 2\Delta_{31} \\
& -\frac{4}{9}\left(2 - 3\sin^2\Delta_{31}\right)\Delta^2_{21} \;,
\end{aligned}
\tag{16}
$$

$$
\begin{aligned}
P_{e\mu} &\approx 4|\hat{\kappa}_{12}|^2 - \left(3|\hat{\kappa}_{12}|^2 - |\hat{\kappa}_{13}|^2 + 2\mathrm{Re}[\hat{\kappa}_{12}\hat{\kappa}^*_{13}]\right)\sin^2\Delta_{31} + \frac{4}{9}\Delta^2_{21} \\
&+\frac{2}{3}\mathrm{Re}[\hat{\kappa}_{12}+\hat{\kappa}_{13}]\Delta_{21}\sin 2\Delta_{31} + 2\mathrm{Im}[\hat{\kappa}_{12}\hat{\kappa}^*_{13}]\sin 2\Delta_{31} \\
&+\frac{4}{3}\left(2\mathrm{Im}[\hat{\kappa}_{12}] - \mathrm{Im}[\hat{\kappa}_{12}+\hat{\kappa}_{13}]\sin^2\Delta_{31}\right)\Delta_{21}\ , \qquad (17) \\
P_{e\tau} &\approx 4|\hat{\kappa}_{13}|^2 - \left(3|\hat{\kappa}_{13}|^2 - |\hat{\kappa}_{12}|^2 + 2\mathrm{Re}[\hat{\kappa}_{12}\hat{\kappa}^*_{13}]\right)\sin^2\Delta_{31} + \frac{4}{9}\Delta^2_{21} \\
&-\frac{2}{3}\mathrm{Re}[\hat{\kappa}_{12}+\hat{\kappa}_{13}]\Delta_{21}\sin 2\Delta_{31} - 2\mathrm{Im}[\hat{\kappa}_{12}\hat{\kappa}^*_{13}]\sin 2\Delta_{31} \\
&-\frac{4}{3}\left(2\mathrm{Im}[\hat{\kappa}_{13}] - \mathrm{Im}[\hat{\kappa}_{12}+\hat{\kappa}_{13}]\sin^2\Delta_{31}\right)\Delta_{21}\ , \qquad (18) \\
P_{\mu\tau} &\approx 4|\hat{\kappa}_{23}|^2 + \left[1 - |\hat{\kappa}_{12}+\hat{\kappa}_{13}|^2 - 4\left(|\hat{\kappa}_{23}|^2 + (\mathrm{Im}[\hat{\kappa}_{23}])^2\right)\right]\sin^2\Delta_{31} \\
&-\frac{2}{3}\left(1+\mathrm{Re}[\hat{\kappa}_{12}-\hat{\kappa}_{13}]\right)\Delta_{21}\sin 2\Delta_{31} + \frac{4}{9}\left(1-3\sin^2\Delta_{31}\right)\Delta^2_{21} \\
&+\left[(2-\epsilon_b-\epsilon_c)\,\mathrm{Im}[\hat{\kappa}_{23}] - \mathrm{Im}[\hat{\kappa}_{12}\hat{\kappa}^*_{13}]\right]\sin 2\Delta_{31} \\
&-\frac{4}{3}\left(2\mathrm{Im}[\hat{\kappa}_{23}] - \mathrm{Im}[\hat{\kappa}_{12}+\hat{\kappa}_{13}+\hat{\kappa}_{23}]\sin^2\Delta_{31}\right)\Delta_{21}\ . \qquad (19)
\end{aligned}
$$

The first term in each of the above six equations is the "zero-distance" term. The last two terms in Eq. (17), (18) or (19) are the "CP-violating" terms.

Suppose that the absolute values of those non-unitary parameters are around their upper bounds, i.e., $|\hat{\kappa}_{12}| \sim 3.5\times10^{-5}$, $|\hat{\kappa}_{13}| \sim 8.0\times10^{-3}$, $|\hat{\kappa}_{23}| \sim 5.0\times10^{-3}$, $|\epsilon_a| \sim 5.5\times10^{-3}$, $|\epsilon_b| \sim 5.0\times10^{-3}$ and $|\epsilon_c| \sim 5.0\times10^{-3}$. We notice that P_{ee} is only sensitive to $|V_{e3}| = |\hat{\kappa}_{12}+\hat{\kappa}_{13}|/\sqrt{2}$, therefore we are able to determine $|\hat{\kappa}_{13}|$ through the detection of $\nu_e \to \nu_e$ oscillation. By measuring the probability of $\nu_\mu \to \nu_\mu$ oscillation it is possible to determine or constrain the value of $\mathrm{Re}[\hat{\kappa}_{23}]$. If the small difference between $P_{\tau\tau}$ and $P_{\mu\mu}$ can be well measured, then we are able to obtain the information on $\mathrm{Re}[\hat{\kappa}_{13}]$. Combined with the value of $|\hat{\kappa}_{13}|$, $\arg(\hat{\kappa}_{13})$ can be determined. As for $\mathrm{Im}[\hat{\kappa}_{23}]$, the most effective way is to probe the CP-violating terms in the $\nu_\mu \to \nu_\tau$ channel.

III. NEUTRINO OSCILLATIONS IN MATTER

When a neutrino beam passes through matter, only ν_e can interact with electrons in the medium via the charged-current interactions, while ν_e, ν_μ and ν_τ can all interact with electrons, protons and neutrons in the medium via the neutral-current interactions. The coherent forward scattering from the constituents of matter modifies the evolution behaviors of the neutrino beam. In the vacuum mass eigenbasis, the evolution equation of neutrinos

can be written as

$$i\frac{d}{dt}|\nu_m(t)\rangle = \tilde{\mathcal{H}}|\nu_m(t)\rangle \ . \tag{20}$$

We use tildes to denote the quantities in matter. For the propagation of neutrinos in matter of constant density, the Hamiltonian $\tilde{\mathcal{H}}$ is given by [16]

$$\tilde{\mathcal{H}} = E + V^T A V^* \ , \tag{21}$$

where $E \equiv \mathrm{diag}(E_1, E_2, E_3)$ is the energy matrix in the mass eigenbasis in vacuum, $A \equiv \mathrm{diag}(V_{CC} - V_{NC}, -V_{NC}, -V_{NC})$, with $V_{CC} \equiv \sqrt{2}G_F n_e$ and $V_{NC} \equiv \frac{1}{\sqrt{2}}G_F n_n$ (n_e and n_n are the electron and neutron densities, respectively). Here V is just the non-unitarity mixing matrix in Eq. (6).

The Hermitian matrix $\tilde{\mathcal{H}}$ can be diagonalized by a unitary transformation $\tilde{\mathcal{H}} = U\tilde{E}U^\dagger$, where $\tilde{E} \equiv \mathrm{diag}(\tilde{E}_1, \tilde{E}_2, \tilde{E}_3)$ is the effective energy matrix in matter. The solution to Eq. (20) can be expressed as

$$|\nu_m(L)\rangle = Ue^{-i\tilde{E}L}U^\dagger|\nu_m(0)\rangle \ , \tag{22}$$

where we have inserted $L = t$. From Eq. (22) we can work out the neutrino oscillation probabilities in matter:

$$\begin{aligned}
\tilde{P}_{\alpha\beta} &= \frac{\left|\left(V^* U e^{-i\tilde{E}L} U^\dagger V^T\right)_{\alpha\beta}\right|^2}{\left(VV^\dagger\right)_{\alpha\alpha}\left(VV^\dagger\right)_{\beta\beta}} = \frac{\left|\left(X^* e^{-i\tilde{E}L} X^T\right)_{\alpha\beta}\right|^2}{\left(XX^\dagger\right)_{\alpha\alpha}\left(XX^\dagger\right)_{\beta\beta}} \\
&= \frac{1}{\left(XX^\dagger\right)_{\alpha\alpha}\left(XX^\dagger\right)_{\beta\beta}}\Big[\left|\left(XX^\dagger\right)_{\alpha\beta}\right|^2 - 4\tilde{A}^{21}_{\alpha\beta}\sin^2\tilde{\Delta}_{21} - 4\tilde{A}^{31}_{\alpha\beta}\sin^2\tilde{\Delta}_{31} - 4\tilde{A}^{32}_{\alpha\beta}\sin^2\tilde{\Delta}_{32} \\
&\qquad -2\tilde{J}^{21}_{\alpha\beta}\sin 2\tilde{\Delta}_{21} - 2\tilde{J}^{31}_{\alpha\beta}\sin 2\tilde{\Delta}_{31} - 2\tilde{J}^{32}_{\alpha\beta}\sin 2\tilde{\Delta}_{32}\Big] \ ,
\end{aligned} \tag{23}$$

where $X \equiv VU^*$, $\tilde{A}^{ij}_{\alpha\beta} = \mathrm{Re}[X_{\alpha i}X_{\beta j}X^*_{\alpha j}X^*_{\beta i}]$, $\tilde{J}^{ij}_{\alpha\beta} = \mathrm{Im}[X_{\alpha i}X_{\beta j}X^*_{\alpha j}X^*_{\beta i}]$ and $\tilde{\Delta}_{ij} \equiv \frac{\tilde{E}_i - \tilde{E}_j}{2}$. Comparing Eq. (23) with Eq. (11), we find that the matrix X, which is also non-unitary, can be regarded as the effective neutrino mixing matrix in matter.

In Appendix A, we present the details of the approximate diagonalization of the Hamiltonian $\tilde{\mathcal{H}}$ by using the perturbation theory. In the results to be presented below, those terms in proportion to $(V_{CC} - 2V_{NC})$ will be neglected. The reason is simple: for ordinary earth matter, which is electrically neutral, we have $n_e \approx n_n$ to a good degree of accuracy, and thus we can safely set $V_{CC} - 2V_{NC} = \sqrt{2}G_F(n_e - n_n) \approx 0$. It is necessary to mention that the subsequent analytical approximations are not very good for a relative large L/E_ν or for

the Δm^2_{21}-dominated oscillation. In addition, we cannot directly obtain Eqs. (14) $\sim$ (19) from Eqs. (27) $\sim$ (32) by setting $V_{CC}, V_{NC} \to 0$. This is because the expansion of $\tilde{\mathcal{H}}$ in Eqs. (A4) $\sim$ (A7) is improper if $V_{CC} = V_{NC} = 0$.

The eigenvalues of $\tilde{\mathcal{H}}$, $\tilde{E}_i$ $(i = 1, 2, 3)$, are related to the effective neutrino masses in matter by the relations $\tilde{E}_i \approx E_\nu + \dfrac{\tilde{\lambda}_i^2}{2E_\nu}$, where E_ν is the energy of neutrinos [2]. Then the effective mass squared differences in matter are given by $\Delta\tilde{\lambda}^2_{ij} \equiv 2E_\nu(\tilde{E}_i - \tilde{E}_j)$ which are shown in Eqs. (24), (25) and (26) to the second order in $\hat{\kappa}_{12}$, $\hat{\kappa}_{13}$, $\hat{\kappa}_{23}$, ϵ_a, ϵ_b, ϵ_c and Δ_{21}.

$$
\begin{aligned}
\Delta\tilde{\lambda}^2_{21} \approx\; & -2E_\nu V_{CC} + \frac{1}{3}\Delta m^2_{21} - 4E_\nu\left(V_{CC} - V_{NC}\right)\epsilon_a \\
& -2E_\nu V_{NC}\left(\epsilon_b + \epsilon_c + 2\mathrm{Re}[\hat{\kappa}_{23}]\right) - \frac{2\left(\Delta m^2_{21}\right)^2}{9E_\nu V_{CC}} \\
& -\frac{\left(2E_\nu V_{NC}\right)^2}{\Delta m^2_{31}}\left[\left(\epsilon_b - \epsilon_c\right)^2 + 4\left(\mathrm{Im}[\hat{\kappa}_{23}]\right)^2\right] \\
& +E_\nu\left(V_{CC} - V_{NC}\right)\left(\left|\hat{\kappa}_{12} - \hat{\kappa}_{13}\right|^2 - 2\epsilon_a^2\right) \\
& -E_\nu V_{NC}\left(\left|\epsilon_b - \hat{\kappa}_{23}\right|^2 + \left|\epsilon_c - \hat{\kappa}_{23}\right|^2 - 2\left|\hat{\kappa}_{12}\right|^2 - 2\left|\hat{\kappa}_{13}\right|^2\right) , \qquad (24)\\
\Delta\tilde{\lambda}^2_{31} \approx\; & \Delta m^2_{31} - 2E_\nu V_{CC} - \frac{1}{3}\Delta m^2_{21} - 4E_\nu\left(V_{CC} - V_{NC}\right)\epsilon_a \\
& -2E_\nu V_{NC}\left(\epsilon_b + \epsilon_c - 2\mathrm{Re}[\hat{\kappa}_{23}]\right) - \frac{\left(\Delta m^2_{21}\right)^2}{9E_\nu V_{CC}} \\
& +\frac{\left(2E_\nu V_{NC}\right)^2}{\Delta m^2_{31}}\left[\left(\epsilon_b - \epsilon_c\right)^2 + 4\left(\mathrm{Im}[\hat{\kappa}_{23}]\right)^2\right] \\
& +E_\nu\left(V_{CC} - V_{NC}\right)\left(\left|\hat{\kappa}_{12} + \hat{\kappa}_{13}\right|^2 - 2\epsilon_a^2\right) \\
& -E_\nu V_{NC}\left(\left|\epsilon_b + \hat{\kappa}_{23}\right|^2 + \left|\epsilon_c + \hat{\kappa}_{23}\right|^2 - 2\left|\hat{\kappa}_{12}\right|^2 - 2\left|\hat{\kappa}_{13}\right|^2\right) , \qquad (25)\\
\Delta\tilde{\lambda}^2_{32} \approx\; & \Delta m^2_{31} - \frac{2}{3}\Delta m^2_{21} + 8E_\nu V_{NC}\mathrm{Re}[\hat{\kappa}_{23}] + \frac{\left(\Delta m^2_{21}\right)^2}{9E_\nu V_{CC}} \\
& +\frac{2\left(2E_\nu V_{NC}\right)^2}{\Delta m^2_{31}}\left[\left(\epsilon_b - \epsilon_c\right)^2 + 4\left(\mathrm{Im}[\hat{\kappa}_{23}]\right)^2\right] \\
& +4E_\nu\left(V_{CC} - V_{NC}\right)\mathrm{Re}[\hat{\kappa}_{12}\hat{\kappa}^*_{13}] - 4E_\nu V_{NC}\left(\epsilon_b + \epsilon_c\right)\mathrm{Re}[\hat{\kappa}_{23}] . \qquad (26)
\end{aligned}
$$

In Eqs. (27) to (32), we present the expansion forms of Eq. (22) for all six neutrino oscillation probabilities to the second order in $\hat{\kappa}_{12}$, $\hat{\kappa}_{13}$, $\hat{\kappa}_{23}$, ϵ_a, ϵ_b, ϵ_c and Δ_{21} in terms of

---

[2] Here we use the notation $\tilde{\lambda}_i$ and $\Delta\tilde{\lambda}^2_{ij}$ to denote the effective neutrino masses and the mass squared differences in matter instead of $\tilde{m}_i$ and $\Delta\tilde{m}^2_{ij}$, because we did not ordering $\tilde{E}_i$ according to their magnitude and the mass spectrum. After ordering $\tilde{E}_i$, we have $\tilde{\lambda}_i = \tilde{m}_i$ and $\Delta\tilde{\lambda}^2_{ij} = \Delta\tilde{m}^2_{ij}$.

the quantities in vacuum.

$$\begin{aligned}
\tilde{P}_{ee} &= 1-2\left|\hat{\kappa}_{12}+\hat{\kappa}_{13}\right|^2\sin^2\left(\Delta_{31}-\frac{V_{CC}L}{2}\right) \\
&\quad -2\left|\frac{4\Delta_{21}}{3V_{CC}L}-\left(\hat{\kappa}_{12}-\hat{\kappa}_{13}\right)\right|^2\sin^2\left(\frac{V_{CC}L}{2}\right)\ ,
\end{aligned} \tag{27}$$

$$\begin{aligned}
\tilde{P}_{\mu\mu} &= 1-\Bigg[1-8\left(\frac{1}{3}\Delta_{21}+2V_{NC}L\mathrm{Re}[\hat{\kappa}_{23}]\right)^2+\frac{2\sqrt{2}\Delta_{21}V_{NC}L}{3\Delta_{31}\left(V_{CC}-2\Delta_{31}\right)}\left(\epsilon_b-\epsilon_c\right) \\
&\quad -4\left(\mathrm{Re}[\hat{\kappa}_{23}]\right)^2+\frac{4V_{CC}L}{\Delta_{31}}\left(\epsilon_b-\epsilon_c\right)\mathrm{Re}[\hat{\kappa}_{23}]-\frac{\left(V_{CC}L\right)^2}{\Delta_{31}}\left(\epsilon_b-\epsilon_c\right)^2\Bigg]\sin^2\Delta_{31} \\
&\quad +\Bigg[2\left(\frac{1}{3}\Delta_{21}+2V_{NC}L\mathrm{Re}[\hat{\kappa}_{23}]\right)-\frac{4\Delta_{21}^2}{9V_{CC}L}-2\left(V_{CC}-V_{NC}\right)L\mathrm{Re}[\hat{\kappa}_{12}\hat{\kappa}_{13}^*] \\
&\quad +2V_{NC}L\left(\epsilon_b+\epsilon_c\right)\mathrm{Re}[\hat{\kappa}_{23}]-\frac{\left(V_{NC}L\right)^2}{\Delta_{31}}\left[\left(\epsilon_b-\epsilon_c\right)^2+4\left(\mathrm{Im}[\hat{\kappa}_{23}]\right)^2\right]\Bigg]\sin 2\Delta_{31} \\
&\quad +2\left|\frac{2\Delta_{21}}{3V_{CC}L}+\hat{\kappa}_{12}\right|^2\cdot\left[2\sin^2\Delta_{31}\sin^2\left(\frac{V_{CC}L}{2}\right)+\frac{1}{2}\sin 2\Delta_{31}\sin\left(V_{CC}L\right)\right] \\
&\quad -4\left|\frac{2\Delta_{21}}{3V_{CC}L}+\hat{\kappa}_{12}\right|^2\sin^2\left(\frac{V_{CC}L}{2}\right)-4\left(\frac{1}{3}\Delta_{21}+2V_{NC}L\mathrm{Re}[\hat{\kappa}_{23}]\right)^2\ ,
\end{aligned} \tag{28}$$

$$\begin{aligned}
\tilde{P}_{\tau\tau} &= 1-\Bigg[1-8\left(\frac{1}{3}\Delta_{21}+2V_{NC}L\mathrm{Re}[\hat{\kappa}_{23}]\right)^2-\frac{2\sqrt{2}\Delta_{21}V_{NC}L}{3\Delta_{31}\left(V_{CC}-2\Delta_{31}\right)}\left(\epsilon_b-\epsilon_c\right) \\
&\quad -4\left(\mathrm{Re}[\hat{\kappa}_{23}]\right)^2-\frac{4V_{CC}L}{\Delta_{31}}\left(\epsilon_b-\epsilon_c\right)\mathrm{Re}[\hat{\kappa}_{23}]-\frac{\left(V_{CC}L\right)^2}{\Delta_{31}}\left(\epsilon_b-\epsilon_c\right)^2\Bigg]\sin^2\Delta_{31} \\
&\quad +\Bigg[2\left(\frac{1}{3}\Delta_{21}+2V_{NC}L\mathrm{Re}[\hat{\kappa}_{23}]\right)-\frac{4\Delta_{21}^2}{9V_{CC}L}-2\left(V_{CC}-V_{NC}\right)L\mathrm{Re}[\hat{\kappa}_{12}\hat{\kappa}_{13}^*] \\
&\quad +2V_{NC}L\left(\epsilon_b+\epsilon_c\right)\mathrm{Re}[\hat{\kappa}_{23}]-\frac{\left(V_{CC}L\right)^2}{\Delta_{31}}\left[\left(\epsilon_b-\epsilon_c\right)^2+4\left(\mathrm{Im}[\hat{\kappa}_{23}]\right)^2\right]\Bigg]\sin 2\Delta_{31} \\
&\quad +2\left|\frac{2\Delta_{21}}{3V_{CC}L}-\hat{\kappa}_{13}\right|^2\cdot\left[2\sin^2\Delta_{31}\sin^2\left(\frac{V_{CC}L}{2}\right)+\frac{1}{2}\sin 2\Delta_{31}\sin\left(V_{CC}L\right)\right] \\
&\quad -4\left|\frac{2\Delta_{21}}{3V_{CC}L}-\hat{\kappa}_{13}\right|^2\sin^2\left(\frac{V_{CC}L}{2}\right)-4\left(\frac{1}{3}\Delta_{21}+2V_{NC}L\mathrm{Re}[\hat{\kappa}_{23}]\right)^2\ ,
\end{aligned} \tag{29}$$

$$\begin{aligned}
\tilde{P}_{e\mu} &= 4|\hat{\kappa}_{12}|^2-\left(3|\hat{\kappa}_{12}|^2-|\hat{\kappa}_{13}|^2+2\mathrm{Re}[\hat{\kappa}_{12}\hat{\kappa}_{13}^*]\right)\sin^2\Delta_{31} \\
&\quad +4\left(\frac{2\Delta_{21}}{3V_{CC}L}+\mathrm{Re}[\hat{\kappa}_{12}]\right)\mathrm{Re}[\hat{\kappa}_{12}+\hat{\kappa}_{13}]\sin^2\Delta_{31}\sin^2\left(\frac{V_{CC}L}{2}\right) \\
&\quad +\left(\frac{2\Delta_{21}}{3V_{CC}L}+\mathrm{Re}[\hat{\kappa}_{12}]\right)\mathrm{Re}[\hat{\kappa}_{12}+\hat{\kappa}_{13}]\sin 2\Delta_{31}\sin\left(V_{CC}L\right)
\end{aligned}$$

$$
\begin{aligned}
&+4\left[\left(\frac{2\Delta_{21}}{3V_{CC}L}\right)^2 - |\hat{\kappa}_{12}|^2\right]\sin^2\left(\frac{V_{CC}L}{2}\right) \\
&\underline{+2\left(\frac{2\Delta_{21}}{3V_{CC}L}\mathrm{Im}[\hat{\kappa}_{12}+\hat{\kappa}_{13}] - \mathrm{Im}[\hat{\kappa}_{12}\hat{\kappa}_{13}^*]\right)\sin 2\Delta_{31}\sin^2\left(\frac{V_{CC}L}{2}\right)} \\
&\underline{+2\left(\frac{2\Delta_{21}}{3V_{CC}L}\mathrm{Im}[\hat{\kappa}_{12}+\hat{\kappa}_{13}] - \mathrm{Im}[\hat{\kappa}_{12}\hat{\kappa}_{13}^*]\right) + \sin^2\Delta_{31}\sin\left(V_{CC}L\right)} \\
&\underline{+2\mathrm{Im}[\hat{\kappa}_{12}\hat{\kappa}_{13}^*]\sin 2\Delta_{31} + \frac{8\Delta_{21}}{3V_{CC}L}\mathrm{Im}[\hat{\kappa}_{12}]\sin\left(V_{CC}L\right)}\ ,
\end{aligned}
\tag{30}
$$

$$
\begin{aligned}
\tilde{P}_{e\tau} \;=\;& 4|\hat{\kappa}_{13}|^2 - \left(3|\hat{\kappa}_{13}|^2 - |\hat{\kappa}_{12}|^2 + 2\mathrm{Re}[\hat{\kappa}_{12}\hat{\kappa}_{13}^*]\right)\sin^2\Delta_{31} \\
&-4\left(\frac{2\Delta_{21}}{3V_{CC}L} - \mathrm{Re}[\hat{\kappa}_{13}]\right)\mathrm{Re}[\hat{\kappa}_{12}+\hat{\kappa}_{13}]\sin^2\Delta_{31}\sin^2\left(\frac{V_{CC}L}{2}\right) \\
&-\left(\frac{2\Delta_{21}}{3V_{CC}L} - \mathrm{Re}[\hat{\kappa}_{13}]\right)\mathrm{Re}[\hat{\kappa}_{12}+\hat{\kappa}_{13}]\sin 2\Delta_{31}\sin\left(V_{CC}L\right) \\
&+4\left[\left(\frac{2\Delta_{21}}{3V_{CC}L}\right)^2 - |\hat{\kappa}_{13}|^2\right]\sin^2\left(\frac{V_{CC}L}{2}\right) \\
&\underline{-2\left(\frac{2\Delta_{21}}{3V_{CC}L}\mathrm{Im}[\hat{\kappa}_{12}+\hat{\kappa}_{13}] - \mathrm{Im}[\hat{\kappa}_{12}\hat{\kappa}_{13}^*]\right)\sin 2\Delta_{31}\sin^2\left(\frac{V_{CC}L}{2}\right)} \\
&\underline{-2\left(\frac{2\Delta_{21}}{3V_{CC}L}\mathrm{Im}[\hat{\kappa}_{12}+\hat{\kappa}_{13}] - \mathrm{Im}[\hat{\kappa}_{12}\hat{\kappa}_{13}^*]\right)\sin^2\Delta_{31}\sin\left(V_{CC}L\right)} \\
&\underline{-2\mathrm{Im}[\hat{\kappa}_{12}\hat{\kappa}_{13}^*]\sin 2\Delta_{31} - \frac{8\Delta_{21}}{3V_{CC}L}\mathrm{Im}[\hat{\kappa}_{13}]\sin\left(V_{CC}L\right)}\ ,
\end{aligned}
\tag{31}
$$

$$
\begin{aligned}
\tilde{P}_{\mu\tau} \;=\;& 4|\hat{\kappa}_{23}|^2 + \Bigg[1 + 4\left(\frac{1}{3}\Delta_{21} + 2V_{NC}L\mathrm{Re}[\hat{\kappa}_{23}]\right)^2 - |\hat{\kappa}_{12}+\hat{\kappa}_{13}|^2 \\
&-4|\hat{\kappa}_{23}|^2 - 4\left(\mathrm{Im}[\hat{\kappa}_{23}]\right)^2 - \frac{\left(V_{CC}L\right)^2}{\Delta_{31}}\left(\epsilon_b - \epsilon_c\right)^2\Bigg]\sin^2\Delta_{31} \\
&-\Bigg[2\left(\frac{1}{3}\Delta_{21} + 2V_{NC}L\mathrm{Re}[\hat{\kappa}_{23}]\right) + \frac{8\Delta_{21}^2}{9V_{CC}L} - 2V_{NC}L\left(\epsilon_b+\epsilon_c\right)\mathrm{Re}[\hat{\kappa}_{23}] \\
&+2\left(V_{CC}-V_{NC}\right)L\mathrm{Re}[\hat{\kappa}_{12}\hat{\kappa}_{13}^*] + \frac{\left(V_{CC}L\right)^2}{\Delta_{31}}\left[\left(\epsilon_b-\epsilon_c\right)^2 + 4\left(\mathrm{Im}[\hat{\kappa}_{23}]\right)^2\right]\Bigg]\sin 2\Delta_{31} \\
&-2\left(\frac{4\Delta_{21}^2}{9\left(V_{CC}L\right)^2} + \frac{2\Delta_{21}}{3V_{CC}L}\mathrm{Re}[\hat{\kappa}_{12}-\hat{\kappa}_{13}] - \mathrm{Re}[\hat{\kappa}_{12}\hat{\kappa}_{13}^*]\right) \\
&\cdot\left[2\sin^2\Delta_{31}\sin^2\left(\frac{V_{CC}L}{2}\right) + \frac{1}{2}\sin 2\Delta_{31}\sin\left(V_{CC}L\right)\right] \\
&+4\left(\frac{1}{3}\Delta_{21} + 2V_{CC}L\mathrm{Re}[\hat{\kappa}_{23}]\right)^2 \underline{+\Bigg[\Bigg(2 - \epsilon_b - \epsilon_c - \frac{4\left(V_{NC}L\right)^2}{\Delta_{31}}\mathrm{Re}[\hat{\kappa}_{23}]}
\end{aligned}
$$

$$
\begin{aligned}
&\underline{\left. -\frac{2\sqrt{2}\Delta_{21}V_{NC}L}{3\Delta_{31}\left(V_{CC}-2\Delta_{31}\right)}\right)\mathrm{Im}[\hat{\kappa}_{23}]-\mathrm{Im}[\hat{\kappa}_{12}\hat{\kappa}^{*}_{13}]\Bigg]\sin 2\Delta_{31}} \\
&\underline{+2\left(\frac{2\Delta_{21}}{3V_{CC}L}\mathrm{Im}[\hat{\kappa}_{12}+\hat{\kappa}_{13}]-\mathrm{Im}[\hat{\kappa}_{12}\hat{\kappa}^{*}_{13}]\right)\sin 2\Delta_{31}\sin^{2}\left(\frac{V_{CC}L}{2}\right)} \\
&\underline{+2\left(\frac{2\Delta_{21}}{3V_{CC}L}\mathrm{Im}[\hat{\kappa}_{12}+\hat{\kappa}_{13}]-\mathrm{Im}[\hat{\kappa}_{12}\hat{\kappa}^{*}_{13}]\right)\sin^{2}\Delta_{31}\sin\left(V_{CC}L\right)} \\
&\underline{+16\mathrm{Im}[\hat{\kappa}_{23}]\left(\frac{1}{3}\Delta_{21}+2V_{NC}L\mathrm{Re}[\hat{\kappa}_{23}]\right)\sin^{2}\Delta_{31}} \\
&\underline{-8\mathrm{Im}[\hat{\kappa}_{23}]\left(\frac{1}{3}\Delta_{21}+2V_{NC}L\mathrm{Re}[\hat{\kappa}_{23}]\right)}\ . \qquad (32)
\end{aligned}
$$

In order to obtain the probabilities of anti-neutrino oscillations $\bar{\nu}_{\alpha}\to\bar{\nu}_{\beta}$, one needs to simultaneously change the signs of V_{CC}, V_{NC} and the terms underlined in the expressions of $\tilde{P}_{\alpha\beta}$.

Different from the case in vacuum, the terms of $\mathcal{O}(\hat{\kappa}_{ij})$ which can be strongly enhanced by large L appear not only in the expression of $\tilde{P}_{\mu\tau}$ but also in those of $\tilde{P}_{\mu\mu}$ and $\tilde{P}_{\tau\tau}$. If we omit the terms of $\mathcal{O}(\hat{\kappa}^{2}_{ij})$, then we get very concise formulas for $\tilde{P}_{\mu\mu}$, $\tilde{P}_{\tau\tau}$ and $\tilde{P}_{\mu\tau}$:

$$
\begin{aligned}
\tilde{P}_{\mu\mu} &= 1-\sin^{2}\Delta_{31}+2\left(\frac{1}{3}\Delta_{21}+2V_{NC}L\mathrm{Re}[\hat{\kappa}_{23}]\right)\sin 2\Delta_{31}\ , \qquad (33)\\
\tilde{P}_{\tau\tau} &= 1-\sin^{2}\Delta_{31}+2\left(\frac{1}{3}\Delta_{21}+2V_{NC}L\mathrm{Re}[\hat{\kappa}_{23}]\right)\sin 2\Delta_{31}\ , \qquad (34)\\
\tilde{P}_{\mu\tau} &= \sin^{2}\Delta_{31}+2\left(\frac{1}{3}\Delta_{21}+2V_{NC}L\mathrm{Re}[\hat{\kappa}_{23}]\right)\sin 2\Delta_{31}+2\mathrm{Im}[\hat{\kappa}_{23}]\sin 2\Delta_{31}\ . \qquad (35)
\end{aligned}
$$

We carry out a numerical analysis to show the difference between the corrections from the non-unitary parameter $\hat{\kappa}_{23}$ and the corrections from the nonzero θ_{13} to the neutrino oscillation probabilities. We compare between two special cases: Case I, we consider a unitary and nearly tri-bimaximal mixing matrix with nonzero θ_{13}, as V_0 given in Eq. (2) with $\theta_{12}=\arcsin\dfrac{1}{\sqrt{3}}$ and $\theta_{23}=45^{\circ}$; Case II, we consider the non-unitary mixing matrix as shown in Eq. (6). The inputs of our numerical calculations are $\Delta m^{2}_{21}=8.0\times10^{-5}\ \mathrm{eV}^{2}$, $\Delta m^{2}_{31}=\pm\ 2.5\times10^{-3}\ \mathrm{eV}^{2}$ (sign "+" for the normal hierarchy and "−" for the inverted hierarchy), the matter density $\rho=2.7\ \mathrm{g/cm}^{3}$. And in Case I we choose $\theta_{13}=10^{\circ}$, in Case II we choose $\epsilon_{a}=-5.5\times10^{-3}$, $\epsilon_{b}=-5.0\times10^{-3}$, $\epsilon_{c}=-5.0\times10^{-3}$, $\hat{\kappa}_{12}=3.5\times10^{-5}\cdot e^{i\frac{\pi}{4}}$, $\hat{\kappa}_{13}=8.0\times10^{-3}\cdot e^{i\frac{\pi}{4}}$ and $|\hat{\kappa}_{23}|=5.0\times10^{-3}$. Our numerical analysis is independent of our analytical results given above, but it confirms the results of our analytical approximations. As for the analytical approximations for Case I, one may refer to [17], in which there exists

a complete set of series expansion for three-flavor neutrino oscillation probabilities in matter in terms of small θ_{13} and $\alpha \equiv \Delta m^2_{21}/\Delta m^2_{31}$.

Fig. 1 shows the effective mass differences in matter as functions of the neutrino beam energy E_ν in Case I and Case II for both the normal and the inverted hierarchies. We can clearly see that the mass squared differences are strongly magnified if E_ν is large (or equivalently if the matter density is large). An interesting point is that in Case II the effective mass difference $\Delta\tilde{m}^2_{32}$ can reach zero at around $E_\nu = 12$ GeV in the normal hierarchy case while in Case I the nonzero $\sin\theta_{13}$ ensures the nonzero value of $\Delta\tilde{m}^2_{32}$. If we choose a nonzero θ_{13} for Case II, we will get similar curves as those of Case I. We find that although the non-unitary parameter $\frac{1}{\sqrt{2}}(\hat{\kappa}_{12}+\hat{\kappa}_{13})$ plays a very similar role as $\sin\theta_{13}e^{-i\delta}$ in the expressions of neutrino oscillation probabilities in vacuum, it has very different effects from $\sin\theta_{13}e^{-i\delta}$ in matter. This finding provides us with a possibility of distinguishing the nonzero θ_{13} from the non-unitary parameters in V.

Taking account of Eqs. (27) to (29) in Ref. [17], we can easily see that the Dirac phase δ dose not appear in the expressions of the eigenvalues of $\tilde{\mathcal{H}}$. However, in the non-unitary case, all the effective mass squared differences contain the terms proportional to $E_\nu V_{NC}\mathrm{Re}[\hat{\kappa}_{23}]$ which is relevant to $\arg(\hat{\kappa}_{23})$ for fixed $|\hat{\kappa}_{23}|$. Fig. 2 shows the effective mass differences in matter as functions of the Dirac phase δ in Case I or the phase of $\hat{\kappa}_{23}$ in Case II for both mass hierarchies, where we have chosen $E_\nu = 50$ GeV. We find that the correction from the term $E_\nu V_{NC}\mathrm{Re}[\hat{\kappa}_{23}]$ can be around 10^{-5} eV^2 in this situation.

Fig. 3 tells us how the probabilities of neutrino oscillations $\nu_\mu \to \nu_\mu$ and $\nu_\mu \to \nu_\tau$ in matter are modified with the changing of the baseline L, where we have chosen $\delta = \frac{\pi}{4}$ and $\arg(\hat{\kappa}_{23}) = \frac{\pi}{4}$. We can clearly see from the figure that the probability $\tilde{P}_{\mu\tau}$ ($\tilde{P}_{\mu\mu}$) can be largely enhanced (depressed) by a long baseline if the matter effect is taken into account. At the baseline $L = 4000$ km, $\tilde{P}_{\mu\tau}$ is about 10^{-3}. Fig. 4 shows the probability $\tilde{P}_{\mu\tau}$ and $\tilde{P}_{\mu\mu}$ as functions of the Dirac phase δ in Case I or the phase of $\hat{\kappa}_{23}$ in Case II for both mass hierarchies with the baselines $L = 1000$ km and $L = 4000$ km. We find that if $\tilde{P}_{\mu\tau}$ can be measured to the level of 10^{-4} at 4000 km from the source and $|\hat{\kappa}_{23}|$ can be well measured, $\arg(\hat{\kappa}_{23})$ may convincingly be determined. Another point worthwhile to point out is that from Eqs. (33) to (35) we find that these three probabilities have approximate sign$[\Delta m^2_{31}]$ - $\arg(\hat{\kappa}_{23})$ degeneracy, which means $\tilde{P}_{\mu\mu,\ \tau\tau,\ \mu\tau}(\Delta m^2_{31}, \arg(\hat{\kappa}_{23})) \approx \tilde{P}_{\mu\mu,\ \tau\tau,\ \mu\tau}(-\Delta m^2_{31}, -\arg(\hat{\kappa}_{23}))$. This

degeneracy is broken by the term $\frac{2}{3}\Delta_{21}\sin 2\Delta_{31}$, which increases with the increase of the baseline L. Fig. 4 shows that this degeneracy breaking can reach 10^{-3} at $L = 4000$ km if the energy of neutrinos is taken to be $E_\nu = 50$ GeV.

We admit that it is very difficult to measure the transition probabilities to the accuracy of 10^{-3} or even 10^{-4} in the present or forthcoming neutrino oscillation experiments. Given the small effects of unitarity violation, however, our numerical results can at least serve to illustrate how sensitive an ambitious long-baseline neutrino experiment should be to this kind of new physics. It is worth remarking two positive aspects of searching for the non-unitarity of V in the ν_μ disappearance or ν_τ appearance experiments. First, the signatures of the non-unitarity can be strongly enhanced by the matter effects, and thus a high energy and a very long baseline are essential to detect appreciable effects of the non-unitarity in the neutrino oscillation experiments. As for neutrinos of energy around 50 GeV, a baseline much longer than 4000 km may have much better sensitivity to the non-unitary parameters in the neutrino mixing matrix, in which case the varying of the terrestrial matter density need to be taken into account [18]. Second, the dependency of the non-unitary parameters $\hat{\kappa}_{ij}$ on the energy of the neutrino beam is different from other neutrino mixing parameters, and thus measuring the energy dependency of the transition probabilities will help to identify the small effects of the non-unitarity.

IV. CONSTRUCTING "DEFORMED UNITARITY TRIANGLES"

In the unitary limit, the neutrino mixing matrix V which relates the neutrino mass eigenstates (ν_1, ν_2, ν_3) to the neutrino flavor eigenstates $(\nu_e, \nu_\mu, \nu_\tau)$ is unitary. The unitarity implies

$$\begin{aligned}
\Delta_\tau: &\quad V_{e1}V^*_{\mu 1} + V_{e2}V^*_{\mu 2} + V_{e3}V^*_{\mu 3} \; = \; 0 \; , \\
\Delta_\mu: &\quad V_{e1}V^*_{\tau 1} + V_{e2}V^*_{\tau 2} + V_{e3}V^*_{\tau 3} \; = \; 0 \; , \\
\Delta_e: &\quad V_{\mu 1}V^*_{\tau 1} + V_{\mu 2}V^*_{\tau 2} + V_{\mu 3}V^*_{\tau 3} \; = \; 0 \; , \\
\Delta_3: &\quad V_{e1}V^*_{e2} + V_{\mu 1}V^*_{\mu 2} + V_{\tau 1}V^*_{\tau 2} \; = \; 0 \; , \\
\Delta_2: &\quad V_{e1}V^*_{e3} + V_{\mu 1}V^*_{\mu 3} + V_{\tau 1}V^*_{\tau 3} \; = \; 0 \; , \\
\Delta_1: &\quad V_{e2}V^*_{e3} + V_{\mu 2}V^*_{\mu 3} + V_{\tau 2}V^*_{\tau 3} \; = \; 0 \; .
\end{aligned} \tag{36}$$

In the complex plane, Eq. (36) corresponds to six unitarity triangles [19] denoted as Δ_τ, Δ_μ, Δ_e, Δ_3, Δ_2 and Δ_1 respectively. The area of each unitarity triangle is $\frac{1}{2}|\mathcal{J}|$, where $\mathcal{J}$ is the Jarlskog invariant measure of CP violation for the unitary MNS mixing matrix. If there is no CP violation (e.g., the tri-bimaximal mixing), the unitarity triangles shrink to segments. In other words, introducing the "unitarity triangles" provides a geometric way to describe CP violation (which can be determined by the area of each triangle) by measuring the CP-conserving quantities (the sides of the triangles).

If the mixing matrix V is non-unitary, the orthogonal relations in Eq. (36) are in general not satisfied and the unitarity triangles in the complex plane are in general open. Note that every two vector sides can determine one triangle in the complex plane, of which twice the area corresponds to one of the nine Jarlskog invariants. To be explicit, the Jarlskog invariant $J^{ij}_{\alpha\beta}$ equals to twice the area of the triangle determined by $V_{\alpha i}V^*_{\beta i}$ and $V_{\alpha j}V^*_{\beta j}$ (or $V_{\alpha i}V^*_{\alpha j}$ and $V_{\beta i}V^*_{\beta j}$). Apparently, if all the six triangles are closed, these nine Jarlskog invariants are all equal (to $\mathcal{J}$).

As for our special scenario, the CP-violating effects are attributed to the phases of $\hat{\kappa}_{12}$, $\hat{\kappa}_{13}$ and $\hat{\kappa}_{23}$. Thus we are able to determine those unitarity-violating parameters by constructing those "deformed unitarity triangles". In Appendix B, we give all the eighteen sides of the six deformed triangles. Table I shows the ratios of two sides of Δ_τ, Δ_μ and Δ_e to the first order of $\hat{\kappa}_{13}$ and $\hat{\kappa}_{23}$. Here we omit the smallest parameter $\hat{\kappa}_{12}$.

One can find from Appendix B that for Δ_τ we have $S_1 + S_2 + S_3 \approx 2\hat{\kappa}_{12} \approx 0$, therefore Δ_τ is almost closed. From Table I we find that in our special scenario, if the ratio $|S_3|/|S_1|$ of Δ_τ can be well measured, $|\hat{\kappa}_{13}|$ can be determined. Combined with the ratio $|S_2|/|S_1|$, the phase of $\hat{\kappa}_{13}$ can also be obtained. The results can be and should be checked by the measurements of $|S_3|/|S_1|$ and $|S_2|/|S_1|$ of Δ_μ, which is a validation of our scenario. In addition, constructing the triangle Δ_e can give us the correlation between $|\hat{\kappa}_{23}|$ and $\arg(\hat{\kappa}_{23})$ if $\hat{\kappa}_{13}$ is fixed.

V. FURTHER DISCUSSIONS

In this section we would like to present a short discussion about the well-accepted see-saw mechanism [6], of which the unitary violation of the mixing matrix V is a general consequence. We show here how to incorporate our parametrization into a generic type-II

TABLE I: Ratios of the absolute values of the sides of Δ_τ, Δ_μ and Δ_e, to the first order in $\hat{\kappa}_{13}$ and $\hat{\kappa}_{23}$. The smallest parameter $\hat{\kappa}_{12}$ is omitted.

Δ_τ	$\dfrac{\lvert S_2\rvert}{\lvert S_1\rvert} = \left\lvert \dfrac{V_{e2}V^*_{\mu 2}}{V_{e1}V^*_{\mu 1}} \right\rvert \approx \lvert 1 - \dfrac{3}{2}\hat{\kappa}_{13}\rvert$
	$\dfrac{\lvert S_3\rvert}{\lvert S_1\rvert} = \left\lvert \dfrac{V_{e3}V^*_{\mu 3}}{V_{e1}V^*_{\mu 1}} \right\rvert \approx \dfrac{\lvert S_3\rvert}{\lvert S_2\rvert} = \left\lvert \dfrac{V_{e3}V^*_{\mu 3}}{V_{e1}V^*_{\mu 1}} \right\rvert \approx \dfrac{3}{2}\lvert\hat{\kappa}_{13}\rvert$
Δ_μ	$\dfrac{\lvert S_2\rvert}{\lvert S_1\rvert} = \left\lvert \dfrac{V_{e2}V^*_{\tau 2}}{V_{e1}V^*_{\tau 1}} \right\rvert \approx \lvert 1 - \dfrac{9}{2}\hat{\kappa}_{13}\rvert$
	$\dfrac{\lvert S_3\rvert}{\lvert S_1\rvert} = \left\lvert \dfrac{V_{e3}V^*_{\tau 3}}{V_{e1}V^*_{\tau 1}} \right\rvert \approx \dfrac{\lvert S_3\rvert}{\lvert S_2\rvert} = \left\lvert \dfrac{V_{e3}V^*_{\tau 3}}{V_{e1}V^*_{\tau 1}} \right\rvert \approx \dfrac{3}{2}\lvert\hat{\kappa}_{13}\rvert$
Δ_e	$\dfrac{\lvert S_1\rvert}{\lvert S_3\rvert} = \left\lvert \dfrac{V_{\mu 2}V^*_{\mu 2}}{V_{\mu 1}V^*_{\tau 1}} \right\rvert \approx \dfrac{1}{3}\lvert 1 - 4\hat{\kappa}_{23} + \hat{\kappa}_{13}\rvert$
	$\dfrac{\lvert S_2\rvert}{\lvert S_3\rvert} = \left\lvert \dfrac{V_{\mu 3}V^*_{\mu 3}}{V_{\mu 1}V^*_{\tau 1}} \right\rvert \approx \dfrac{1}{3}\lvert 1 - 4\hat{\kappa}_{23} - \hat{\kappa}_{13}\rvert$
	$\dfrac{\lvert S_2\rvert}{\lvert S_1\rvert} = \left\lvert \dfrac{V_{\mu 3}V^*_{\mu 3}}{V_{\mu 1}V^*_{\tau 1}} \right\rvert \approx 2\lvert 1 - 3\hat{\kappa}_{13}\rvert$

seesaw model. In the type-II seesaw model [21] which contains n right-handed neutrinos, the neutrino mass terms can be written as

$$-\mathcal{L}_{\rm mass} \;=\; \frac{1}{2}\,\overline{\begin{pmatrix}\nu_{\rm L} & N^c_{\rm R}\end{pmatrix}}\begin{pmatrix}M_{\rm L} & M_{\rm D}\\ M^T_{\rm D} & M_{\rm R}\end{pmatrix}\begin{pmatrix}\nu^c_{\rm L}\\ N_{\rm R}\end{pmatrix} \;+\; {\rm h.c.}\;, \tag{37}$$

where $\nu^c_{\rm L} \equiv C\overline{\nu_{\rm L}}^T$ with C being the charge conjugation matrix, likewise for $N^c_{\rm R}$. Here, $M_{\rm L}$ is a 3×3 matrix, $M_{\rm D}$ is a $3\times n$ matrix and $M_{\rm R}$ is a $n \times n$ matrix. The overall $(n+3)\times(n+3)$ neutrino mass matrix in $\mathcal{L}_{\rm mass}$, denoted as $\mathcal{M}$, can be diagonalized by the unitary transformation $\mathcal{U}^\dagger \mathcal{M}\mathcal{U}^* = \widehat{\mathcal{M}}$; or explicitly,

$$\begin{pmatrix}V & R\\ S & U\end{pmatrix}^\dagger \begin{pmatrix}M_{\rm L} & M_{\rm D}\\ M^T_{\rm D} & M_{\rm R}\end{pmatrix}\begin{pmatrix}V & R\\ S & U\end{pmatrix}^* = \begin{pmatrix}\widehat{M}_\nu & \mathbf{0}\\ \mathbf{0} & \widehat{M}_{\rm N}\end{pmatrix}\,, \tag{38}$$

where $\widehat{M}_\nu = {\rm diag}\,(m_1, m_2, m_3)$ and $\widehat{M}_{\rm N} = {\rm diag}\,(M_1, \cdots, M_n)$ with m_i and M_i being the light and heavy Majorana neutrino masses, respectively. Note that the submatrices V, U, R and S are all non-unitary. Suppose the mass eigenstates and the flavor eigenstates of the charged

leptons are identical. In the basis of mass states, the standard charged-current interactions of neutrinos can be written as

$$-\mathcal{L}_{\rm cc} = \frac{g}{\sqrt{2}}\left[\overline{\begin{pmatrix} e & \mu & \tau \end{pmatrix}_{\rm L}}\, V\gamma^\mu \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}_{\rm L} W^-_\mu + \overline{\begin{pmatrix} e & \mu & \tau \end{pmatrix}_{\rm L}}\, R\gamma^\mu \begin{pmatrix} N_1 \\ \vdots \\ N_n \end{pmatrix}_{\rm L} W^-_\mu \right] + \text{h.c.} \,. \quad (39)$$

One can find that the matrix R describes the strength of the charged-current interaction between the charged leptons and the heavy neutrinos. In addition, the deviation of V from unitary is also characterized by R, since the unitarity of $\mathcal{U}$ requires $VV^\dagger + RR^\dagger = \mathbf{1}$ [20].

In fact the diagonalization of $\mathcal{M}$ can be divided into two steps by decomposing $\mathcal{U}$ into the product of two unitary matrices $\mathcal{W}$ and $\mathcal{V}$:

$$\mathcal{V}^\dagger \mathcal{W}^\dagger \begin{pmatrix} M_{\rm L} & M_{\rm D} \\ M_{\rm D}^T & M_{\rm R} \end{pmatrix} \mathcal{W}^* \mathcal{V}^* \equiv \mathcal{V}^\dagger \begin{pmatrix} M_\nu & \mathbf{0} \\ \mathbf{0} & M_{\rm N} \end{pmatrix} \mathcal{V}^* = \begin{pmatrix} \widehat{M}_\nu & \mathbf{0} \\ \mathbf{0} & \widehat{M}_{\rm N} \end{pmatrix} , \quad (40)$$

where $\mathcal{W}$ and $\mathcal{V}$ take the general forms

$$\mathcal{W} = \begin{pmatrix} U_L & B \\ C & U_R \end{pmatrix} , \quad \mathcal{V} = \begin{pmatrix} V_L & \mathbf{0} \\ \mathbf{0} & V_R \end{pmatrix} . \quad (41)$$

The matrices U_L, B, C and U_R are in general non-unitary, but V_L and V_R are unitary matrices. We can easily find that: $V = U_L V_L$, $R = BV_R$. Let's count the degrees of freedom of these matrices. The $(n+3)\times(n+3)$ unitary matrix contains $(n+3)^2$ degrees of freedom. Suppose freedom left for $\mathcal{W}$ is $(n+3)^2 - 3^2 - n^2 = 2\times 3$ the parameters in V_L and V_R are all free, the degrees of $\times n$. An ansatz made for $\mathcal{W}$ in Ref. [22] is to suppose that B is an arbitrary $3\times n$ matrix which contains just $2\times 3n$ degrees of freedom and then parametrize $\mathcal{W}$ as

$$\mathcal{W} = \begin{pmatrix} \sqrt{\mathbf{1} - BB^\dagger} & B \\ -B^\dagger & \sqrt{\mathbf{1} - B^\dagger B} \end{pmatrix} . \quad (42)$$

Comparing Eqs. (38), (40), (41) and (42) with the parametrization $V = H\cdot V_0$ shown in Section I, we can simply choose: $V_0 = V_L$, then we can find that $H = U_L = \sqrt{\mathbf{1} - BB^\dagger} = \sqrt{\mathbf{1} - BV_R V_R^\dagger B^\dagger} = \sqrt{\mathbf{1} - RR^\dagger}$ which contains the same degrees of freedom as the Hermitian matrix in Eq. (1). That is to say in the parametrization we have chosen, the unitarity-violating parameters in H can be expressed as the functions of $RR^\dagger$, where R is of the order

of M_D/M_R. In other words, the deviation from the unitarity which is described by ϵ in Eq. (4) is of the order of $\left(\frac{M_D}{M_R}\right)^2$.

In the canonical seesaw scenarios, the light neutrino masses are attributed to the leading-order contribution $M_\nu = -M_D M_R^{-1} M_D^T$ (type-I seesaw) or $M_\nu = M_L - M_D M_R^{-1} M_D^T$ (type-II seesaw). In order to obtain the light neutrino mass scale $m_\nu \sim 0.1$ eV, the mass scale of right-handed Majorana neutrinos is expected to be as high as $m_R \sim 10^{14}$ GeV. In such a case, one finds $M_D/M_R \sim 10^{-12}$ or equivalently $\epsilon \sim 10^{-24}$, which is too small to be detected. The possible ways out have recently been discussed (see, e.g., Refs. [23] and [24]). For instance, one may first impose certain flavor symmetries on the textures of M_D, M_R and M_L to guarantee $M_D M_R^{-1} M_D^T = 0$ (type-I seesaw) or $M_L - M_D M_R^{-1} M_D^T = 0$ (type-II seesaw) and then introduce slight perturbations to them so as to produce the tiny light neutrino masses. As a consequence, the mass scale of three light neutrinos in this approach is essentially decoupled from the magnitude of R. In the models proposed in Refs. [23, 24], for example, the right-handed Majorana neutrinos are assumed to be around the TeV scale such that $M_D/M_R \sim \mathcal{O}\ (10^{-1})$ holds and the elements of ϵ can be as large as $\sim \mathcal{O}\ (10^{-2})$, just close to their upper bounds constrained by current experimental data. These kinds of seesaw models will be tested at the LHC or ILC [25]; and one of their low-energy consequences, which is just the unitarity violation of V under discussion, will also be tested in the long-baseline neutrino oscillation experiments.

VI. SUMMARY

As we are about to enter an era of high precision neutrino physics, a general and comprehensive analysis of the non-unitary neutrino mixing matrix makes sense and will be useful for phenomenological explanations of future measurements and tests of type-I and type-II seesaw models. In this paper we have investigated a new pattern of the neutrino mixing matrix which is the product of an arbitrary Hermitian matrix and the well-known tri-bimaximal mixing matrix. Starting with this non-unitary mixing matrix, we have presented a complete set of series expansion formulas for neutrino oscillation probabilities both in vacuum and in matter of constant density. We have carried out a numerical analysis to emphasize the importance of matter effects in the measurements of the non-unitary parameters and in distinguishing their effects from the effects induced by small θ_{13}. We find that

measuring the probability of $\nu_\mu \to \nu_\tau$ or $\nu_\mu \to \nu_\mu$ oscillation with large neutrino energy (e.g., ~ 50 GeV) and a relatively long baseline (e.g., several thousand km) is a viable way to detect those non-unitary parameters. We have also discussed the possibility of determining the small non-unitary perturbations and the extra CP-violating phases by constructing the "deformed unitarity triangles". Finally we have shown that our parametrization of the non-unitary mixing matrix can be naturally incorporated into the generic type-II seesaw model.

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APPENDIX A: DIAGONALIZING THE HAMILTONIAN IN MATTER BY USING THE PERTURBATION THEORY

In this appendix we use the perturbation theory to diagonalize the effect Hamiltonian $\tilde{\mathcal{H}} = U\tilde{E}U^\dagger$ in matter of constant density. In the series expansion of $\tilde{\mathcal{H}}$, we regard $\hat{\kappa}_{12}$, $\hat{\kappa}_{13}$, $\hat{\kappa}_{23}$, ϵ_a, ϵ_b, ϵ_c and $\alpha \equiv \Delta m^2_{21}/\Delta m^2_{31}$ as the small parameters of the same order, and perform the diagonalization to the second order of them. Now we are going to diagonalize the effective Hamiltonian $\tilde{\mathcal{H}} = E + V^T A V^*$.

First we draw $E_1 \cdot \mathbf{1}$ (which contributes only a pure phase $e^{-iE_1 L}$ to the oscillation amplitude and nothing to the oscillation probabilities) out of $\tilde{\mathcal{H}}$ and rewrite $\tilde{\mathcal{H}}$ as

$$\tilde{\mathcal{H}} \;=\; E' + V^T A V^* \;=\; \frac{1}{2E_\nu} \cdot \mathrm{diag}\left(0, \alpha, \Delta m^2_{31}\right) + V^T A V^* \;, \tag{A1}$$

where $A \equiv \mathrm{diag}\left(V_{CC} - V_{NC}, -V_{NC}, -V_{NC}\right)$. We find that it is easier to perform the diago-

nalization of $\tilde{\mathcal{H}}' \equiv V_0 \tilde{\mathcal{H}} V_0^\dagger = V_0 E' V_0^\dagger + H^T A H^*$, with

$$V_0 E V_0^\dagger = \frac{\Delta m_{31}^2}{2E_\nu} \left[\frac{1}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} + \frac{\alpha}{3} \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix} \right] , \tag{A2}$$

and

$$\begin{aligned} H^T A H^* &= (\mathbf{1} + \epsilon)^T A \, (\mathbf{1} + \epsilon)^* \\ &= V_{CC} \left[\begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix} + \epsilon^* \begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix} + \begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix} \epsilon^* + \epsilon^* \begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix} \epsilon^* \right] \\ &\quad - V_{NC} \left(2\epsilon^* + \epsilon^{*2} \right) - V_{NC} \cdot \mathbf{1} , \end{aligned} \tag{A3}$$

The term $-V_{NC} \cdot \mathbf{1}$ can be neglected for the same reason as omitting the term $E_1 \cdot \mathbf{1}$. Then we will diagonalize $\tilde{\mathcal{H}}'$ with $\tilde{\mathcal{H}}' = W \tilde{E}' W^\dagger$ using the perturbation theory. We can easily find that $\tilde{E} = \tilde{E}' + E_1 - V_{CC}$ and $U = V_0^\dagger W$. The matrix X, which describes the mixing in matter, can be expressed as $X = V U^* = H V_0 V_0^\dagger W = H W$, which is also non-unitary.

We write

$$\tilde{\mathcal{H}}' = \tilde{\mathcal{H}}'^{(0)} + \tilde{\mathcal{H}}'^{(1)} + \tilde{\mathcal{H}}'^{(2)} , \tag{A4}$$

where

$$\tilde{\mathcal{H}}'^{(0)} = \frac{\Delta m_{31}^2}{4E_\nu} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} + V_{CC} \begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix} , \tag{A5}$$

$$\tilde{\mathcal{H}}'^{(1)} = \frac{1}{3} \cdot \frac{\Delta m_{21}^2}{2E_\nu} \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix} + V_{CC} \begin{pmatrix} 2\epsilon_a & \hat{\kappa}_{12}^* & \hat{\kappa}_{13}^* \\ \hat{\kappa}_{12} & 0 & 0 \\ \hat{\kappa}_{13} & 0 & 0 \end{pmatrix} - 2V_{NC} \begin{pmatrix} \epsilon_a & \hat{\kappa}_{12}^* & \hat{\kappa}_{13}^* \\ \hat{\kappa}_{12} & \epsilon_b & \hat{\kappa}_{23}^* \\ \hat{\kappa}_{13} & \hat{\kappa}_{23} & \epsilon_c \end{pmatrix} , \tag{A6}$$

$$\tilde{\mathcal{H}}'^{(2)} = V_{CC} \begin{pmatrix} \epsilon_a^2 & \epsilon_a \hat{\kappa}_{12}^* & \epsilon_a \hat{\kappa}_{13}^* \\ \epsilon_a \hat{\kappa}_{12} & |\hat{\kappa}_{12}|^2 & \hat{\kappa}_{12} \hat{\kappa}_{13}^* \\ \epsilon_a \hat{\kappa}_{13} & \hat{\kappa}_{12}^* \hat{\kappa}_{13} & |\hat{\kappa}_{13}|^2 \end{pmatrix}$$

$$
-V_{NC}\begin{pmatrix} \epsilon_a^2+|\hat{\kappa}_{12}|^2+|\hat{\kappa}_{13}|^2 & (\epsilon_a+\epsilon_b)\,\hat{\kappa}_{12}^*+\hat{\kappa}_{13}^*\hat{\kappa}_{23} & (\epsilon_a+\epsilon_c)\,\hat{\kappa}_{13}^*+\hat{\kappa}_{12}^*\hat{\kappa}_{23} \\ (\epsilon_a+\epsilon_b)\,\hat{\kappa}_{12}+\hat{\kappa}_{13}\hat{\kappa}_{23}^* & \epsilon_b^2+|\hat{\kappa}_{12}|^2+|\hat{\kappa}_{23}|^2 & (\epsilon_b+\epsilon_c)\,\hat{\kappa}_{23}^*+\hat{\kappa}_{12}\hat{\kappa}_{13}^* \\ (\epsilon_a+\epsilon_c)\,\hat{\kappa}_{13}+\hat{\kappa}_{12}\hat{\kappa}_{23} & (\epsilon_b+\epsilon_c)\,\hat{\kappa}_{23}+\hat{\kappa}_{12}^*\hat{\kappa}_{13} & \epsilon_c^2+|\hat{\kappa}_{13}|^2+|\hat{\kappa}_{23}|^2 \end{pmatrix} . \tag{A7}
$$

For the eigenvalues and the eigenvectors, we also write $\tilde{E'}_i = \tilde{E'}_i^{(0)} + \tilde{E'}_i^{(1)} + \tilde{E'}_i^{(2)}$ and $v_i = v_i^{(0)} + v_i^{(1)} + v_i^{(2)}$ (for $i = 1, 2, 3$). The unitary matrix $W = (v_1, v_2, v_3)$.

$\tilde{\mathcal{H}'}^{(0)}$ can be easily diagonalized. The eigenvalues and the eigenvectors of $\tilde{\mathcal{H}'}^{(0)}$ are:

$$
\tilde{E'}_1^{(0)} = V_{CC}\ , \quad \tilde{E'}_2^{(0)} = 0\ , \quad \tilde{E'}_i^{(0)} = \frac{\Delta m_{31}^2}{2E_\nu}\ ; \tag{A8}
$$

and

$$
v_1^{(0)} = (1,0,0)^T\ , \quad v_2^{(0)} = \left(0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)^T\ , \quad v_3^{(0)} = \left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)^T\ . \tag{A9}
$$

Then, the first and the second order corrections to the eigenvalues are given by

$$
\tilde{E'}_i^{(1)} = \tilde{\mathcal{H}'}_{ii}^{(1)}\ , \tag{A10}
$$

$$
\tilde{E'}_i^{(2)} = \tilde{\mathcal{H}'}_{ii}^{(2)} + \sum_{j\neq i} \frac{\left|\tilde{\mathcal{H}'}_{ji}^{(1)}\right|^2}{\tilde{E'}_i^{(0)} - \tilde{E'}_j^{(0)}}\ ; \tag{A11}
$$

and the corrections to the eigenvectors are calculated by

$$
v_i^{(1)} = \sum_{j\neq i} \frac{\tilde{\mathcal{H}'}_{ji}^{(1)}}{\tilde{E'}_i^{(0)} - \tilde{E'}_j^{(0)}} \cdot v_j^{(0)}\ , \tag{A12}
$$

$$
\begin{aligned}
v_i^{(2)} = {} & \sum_{j\neq i} \left[\frac{\tilde{\mathcal{H}'}_{ji}^{(2)}}{\tilde{E'}_i^{(0)} - \tilde{E'}_j^{(0)}} + \sum_{k\neq i} \frac{\tilde{\mathcal{H}'}_{jk}^{(1)}\tilde{\mathcal{H}'}_{ki}^{(1)}}{\left(\tilde{E'}_i^{(0)} - \tilde{E'}_j^{(0)}\right)\left(\tilde{E'}_i^{(0)} - \tilde{E'}_k^{(0)}\right)} - \frac{\tilde{\mathcal{H}'}_{ii}^{(1)}\tilde{\mathcal{H}'}_{jk}^{(1)}}{\left(\tilde{E'}_i^{(0)} - \tilde{E'}_j^{(0)}\right)^2} \right] \cdot v_j^{(0)} \\
& -\frac{1}{2}\left[\sum_{j\neq i} \frac{\left|\tilde{\mathcal{H}'}_{ji}^{(1)}\right|^2}{\left(\tilde{E'}_i^{(0)} - \tilde{E'}_j^{(0)}\right)^2} \right] \cdot v_i^{(0)}\ ,
\end{aligned} \tag{A13}
$$

where $\tilde{\mathcal{H}'}_{ij}^{(n)} \equiv {v_i^{(0)}}^\dagger \tilde{\mathcal{H}'}^{(n)} v_j^{(0)}$.

Inserting Eqs. (A8), (A9) into Eqs. (A10) and (A12), we obtain

$$
\tilde{E'}_1^{(0)} = \frac{1}{3}\cdot\frac{\Delta m_{21}^2}{2E_\nu} + 2\epsilon_a\left(V_{CC} - V_{NC}\right)\ , \tag{A14}
$$

$$
\tilde{E'}_2^{(0)} = \frac{2}{3}\cdot\frac{\Delta m_{21}^2}{2E_\nu} - V_{NC}\left(\epsilon_b + \epsilon_c + 2\mathrm{Re}[\hat{\kappa}_{23}]\right)\ , \tag{A15}
$$

$$\tilde{E}'^{(0)}_3 = -V_{NC}\left(\epsilon_b + \epsilon_c - 2\mathrm{Re}[\hat{\kappa}_{23}]\right) ; \tag{A16}$$

and

$$W^{(1)} = \begin{pmatrix} 0 & -\dfrac{2\sqrt{2}\Delta_{21}}{3V_{CC}L} & 0 \\ \dfrac{2\Delta_{21}}{3V_{CC}L} & \dfrac{V_{NC}}{2\sqrt{2}\Delta_{31}}\left(\epsilon_b - \epsilon_c + 2\mathrm{Im}[\hat{\kappa}_{23}]\right) & -\dfrac{V_{NC}}{2\sqrt{2}\Delta_{31}}\left(\epsilon_b - \epsilon_c - 2\mathrm{Im}[\hat{\kappa}_{23}]\right) \\ -\dfrac{2\Delta_{21}}{3V_{CC}L} & \dfrac{V_{NC}}{2\sqrt{2}\Delta_{31}}\left(\epsilon_b - \epsilon_c + 2\mathrm{Im}[\hat{\kappa}_{23}]\right) & \dfrac{V_{NC}}{2\sqrt{2}\Delta_{31}}\left(\epsilon_b - \epsilon_c - 2\mathrm{Im}[\hat{\kappa}_{23}]\right) \end{pmatrix} , \tag{A17}$$

where $\Delta_{ij} \equiv \Delta m^2_{ij} L/(4E_\nu)$ with $\Delta m^2_{ij} \equiv m^2_i - m^2_j$ (for $ij = 21, 31, 32$), and the terms proportional to $(V_{CC} - 2V_{NC})$ are omitted.

We can further calculate $X = HW$ to the first order in $\hat{\kappa}_{12}$, $\hat{\kappa}_{13}$, $\hat{\kappa}_{23}$, ϵ_a, ϵ_b, ϵ_c and α:

$$X \approx \begin{pmatrix} 1+\epsilon_a & -\dfrac{2\sqrt{2}\Delta_{21}}{3V_{CC}L} + \dfrac{\hat{\kappa}_{12} - \hat{\kappa}_{13}}{\sqrt{2}} & \dfrac{\hat{\kappa}_{12} + \hat{\kappa}_{13}}{\sqrt{2}} \\ \dfrac{2\Delta_{21}}{3V_{CC}L} + \hat{\kappa}^*_{12} & \dfrac{b - \hat{\kappa}_{23}}{\sqrt{2}} + \dfrac{V_{NC}\left(\epsilon_b - \epsilon_c + 2\mathrm{Im}[\hat{\kappa}_{23}]\right)}{2\sqrt{2}\Delta_{31}} & \dfrac{b + \hat{\kappa}_{23}}{\sqrt{2}} - \dfrac{V_{NC}\left(\epsilon_b - \epsilon_c - 2\mathrm{Im}[\hat{\kappa}_{23}]\right)}{2\sqrt{2}\Delta_{31}} \\ -\dfrac{2\Delta_{21}}{3V_{CC}L} + \hat{\kappa}^*_{13} & -\dfrac{c - \hat{\kappa}_{23}}{\sqrt{2}} + \dfrac{V_{NC}\left(\epsilon_b - \epsilon_c + 2\mathrm{Im}[\hat{\kappa}_{23}]\right)}{2\sqrt{2}\Delta_{31}} & \dfrac{c + \hat{\kappa}_{23}}{\sqrt{2}} + \dfrac{V_{NC}\left(\epsilon_b - \epsilon_c - 2\mathrm{Im}[\hat{\kappa}_{23}]\right)}{2\sqrt{2}\Delta_{31}} \end{pmatrix} . \tag{A18}$$

In this paper we do not order the eigenvalues of $\tilde{\mathcal{H}}$ according to their magnitude and the mass spectrum. This ordering does not change the oscillation probabilities. In this appendix we give the results of $\tilde{E}'^{(0)}_i$, $\tilde{E}'^{(1)}_i$, $W^{(0)}$ and $W^{(1)}$, which are enough for calculating the probabilities P_{ee}, $P_{e\mu}$ and $P_{e\tau}$ to the second order. $\tilde{E}'^{(2)}_i$ and $v^{(2)}_i$, which are not shown here, only correct the terms of the second order in $P_{\mu\mu}$, $P_{\mu\tau}$ and $P_{\tau\tau}$.

APPENDIX B: SIDES OF SIX "DEFORMED UNITARITY TRIANGLES"

Sides of six "deformed unitarity triangles" to the first order in $\hat{\kappa}_{12}$, $\hat{\kappa}_{13}$, $\hat{\kappa}_{23}$, ϵ_a, ϵ_b and ϵ_c.

- Δ_τ:

$$S_1 = V_{e1}V^*_{\mu 1} \approx -\frac{1}{3} - \frac{1}{3}\left(\epsilon_a + \epsilon_b\right) + \frac{1}{6}\left(5\hat{\kappa}_{12} - \hat{\kappa}_{13} + 2\hat{\kappa}^*_{23}\right) , \tag{B1}$$

$$S_2 = V_{e2}V^*_{\mu 2} \approx \frac{1}{3} + \frac{1}{3}\left(\epsilon_a + \epsilon_b\right) + \frac{1}{3}\left(2\hat{\kappa}_{12} - \hat{\kappa}_{13} - \hat{\kappa}^*_{23}\right) , \tag{B2}$$

$$S_3 = V_{e3}V^*_{\mu 3} \approx \frac{1}{2}\left(\hat{\kappa}_{12} + \hat{\kappa}_{13}\right) , \tag{B3}$$

with $S_1 + S_2 + S_3 \approx 2\hat{\kappa}_{12}$.

- Δ_μ:

$$S_1 = V_{e1}V_{\tau1}^* \approx \frac{1}{3}+\frac{1}{3}(\epsilon_a+\epsilon_c)+\frac{1}{6}(5\hat{\kappa}_{13}-\hat{\kappa}_{12}-2\hat{\kappa}_{23}^*) \ , \quad \text{(B4)}$$

$$S_2 = V_{e2}V_{\tau2}^* \approx -\frac{1}{3}-\frac{1}{3}(\epsilon_a+\epsilon_c)+\frac{1}{3}(2\hat{\kappa}_{13}-\hat{\kappa}_{12}+\hat{\kappa}_{23}^*) \ , \quad \text{(B5)}$$

$$S_3 = V_{e3}V_{\tau3}^* \approx \frac{1}{2}(\hat{\kappa}_{12}+\hat{\kappa}_{13}) \ , \quad \text{(B6)}$$

with $S_1+S_2+S_3 \approx 2\hat{\kappa}_{13}$.

- Δ_e:

$$S_1 = V_{\mu1}V_{\tau1}^* \approx -\frac{1}{6}-\frac{1}{6}(\epsilon_b+\epsilon_c)+\frac{1}{3}(\hat{\kappa}_{23}+\hat{\kappa}_{12}^*-\hat{\kappa}_{13}) \ , \quad \text{(B7)}$$

$$S_2 = V_{\mu2}V_{\tau2}^* \approx -\frac{1}{3}-\frac{1}{3}(\epsilon_b+\epsilon_c)+\frac{1}{3}(2\hat{\kappa}_{23}-\hat{\kappa}_{12}^*+\hat{\kappa}_{13}) \ , \quad \text{(B8)}$$

$$S_3 = V_{\mu3}V_{\tau3}^* \approx \frac{1}{2}+\frac{1}{2}(\epsilon_b+\epsilon_c)+\hat{\kappa}_{23} \ , \quad \text{(B9)}$$

with $S_1+S_2+S_3 \approx 2\hat{\kappa}_{23}$.

- Δ_3:

$$S_1 = V_{e1}V_{e2}^* \approx \frac{\sqrt{2}}{3}(1+2\epsilon_a)-\frac{1}{3\sqrt{2}}\left(\mathrm{Re}[\hat{\kappa}_{12}-\hat{\kappa}_{13}]-3i\mathrm{Im}[\hat{\kappa}_{12}-\hat{\kappa}_{13}]\right) \ , \quad \text{(B10)}$$

$$S_2 = V_{\mu1}V_{\mu2}^* \approx -\frac{1}{3\sqrt{2}}(1+2\epsilon_b)+\frac{1}{3\sqrt{2}}\left(\mathrm{Re}[\hat{\kappa}_{12}+2\hat{\kappa}_{23}]-3i\mathrm{Im}[\hat{\kappa}_{12}]\right) \ , \quad \text{(B11)}$$

$$S_3 = V_{\tau1}V_{\tau2}^* \approx -\frac{1}{3\sqrt{2}}(1+2\epsilon_c)-\frac{1}{3\sqrt{2}}\left(\mathrm{Re}[\hat{\kappa}_{13}-2\hat{\kappa}_{23}]-3i\mathrm{Im}[\hat{\kappa}_{13}]\right) \ , \quad \text{(B12)}$$

with $S_1+S_2+S_3 \approx -\frac{\sqrt{2}}{3}(2\epsilon_a-\epsilon_b-\epsilon_c)+\frac{\sqrt{2}}{3}\left(\mathrm{Re}[2\hat{\kappa}_{23}+\hat{\kappa}_{12}-\hat{\kappa}_{13}]-3i\mathrm{Im}[\hat{\kappa}_{12}-\hat{\kappa}_{13}]\right)$.

- Δ_2:

$$S_1 = V_{e1}V_{e3}^* \approx \frac{1}{\sqrt{3}}(\hat{\kappa}_{12}^*+\hat{\kappa}_{13}^*) \ , \quad \text{(B13)}$$

$$S_2 = V_{\mu1}V_{\mu3}^* \approx -\frac{1}{2\sqrt{3}}(1+2\epsilon_b)+\frac{1}{\sqrt{3}}\left(\hat{\kappa}_{12}^*+i\mathrm{Im}[\hat{\kappa}_{23}]\right) \ , \quad \text{(B14)}$$

$$S_3 = V_{\tau1}V_{\tau3}^* \approx \frac{1}{2\sqrt{3}}(1+2\epsilon_c)+\frac{1}{\sqrt{3}}\left(\hat{\kappa}_{13}^*+i\mathrm{Im}[\hat{\kappa}_{23}]\right) \ , \quad \text{(B15)}$$

with $S_1+S_2+S_3 \approx \frac{1}{2\sqrt{3}}(\epsilon_b-\epsilon_c)-\frac{2}{\sqrt{3}}\left(\hat{\kappa}_{12}^*+\hat{\kappa}_{13}^*+i\mathrm{Im}[\hat{\kappa}_{23}]\right)$.

- Δ_1:

$$S_1 = V_{e2}V_{e3}^* \approx \frac{1}{\sqrt{6}}(\hat{\kappa}_{12}^*+\hat{\kappa}_{13}^*) \ , \quad \text{(B16)}$$

$$S_2 \;=\; V_{\mu 2}V^*_{\mu 3} \;\approx\; \frac{1}{\sqrt{6}}\left(1+2\epsilon_b\right) + \frac{1}{\sqrt{6}}\left(\hat{\kappa}^*_{12} - 2i\mathrm{Im}[\hat{\kappa}_{23}]\right) \;, \tag{B17}$$

$$S_3 \;=\; V_{\tau 2}V^*_{\tau 3} \;\approx\; -\frac{1}{\sqrt{6}}\left(1+2\epsilon_c\right) + \frac{1}{\sqrt{6}}\left(\hat{\kappa}^*_{13} - 2i\mathrm{Im}[\hat{\kappa}_{23}]\right) \;, \tag{B18}$$

with $S_1 + S_2 + S_3 \approx -\sqrt{\frac{2}{3}}\left(\epsilon_b - \epsilon_c\right) + \sqrt{\frac{2}{3}}\left(\hat{\kappa}^*_{12} + \hat{\kappa}^*_{13} - 2i\mathrm{Im}[\hat{\kappa}_{23}]\right)$.

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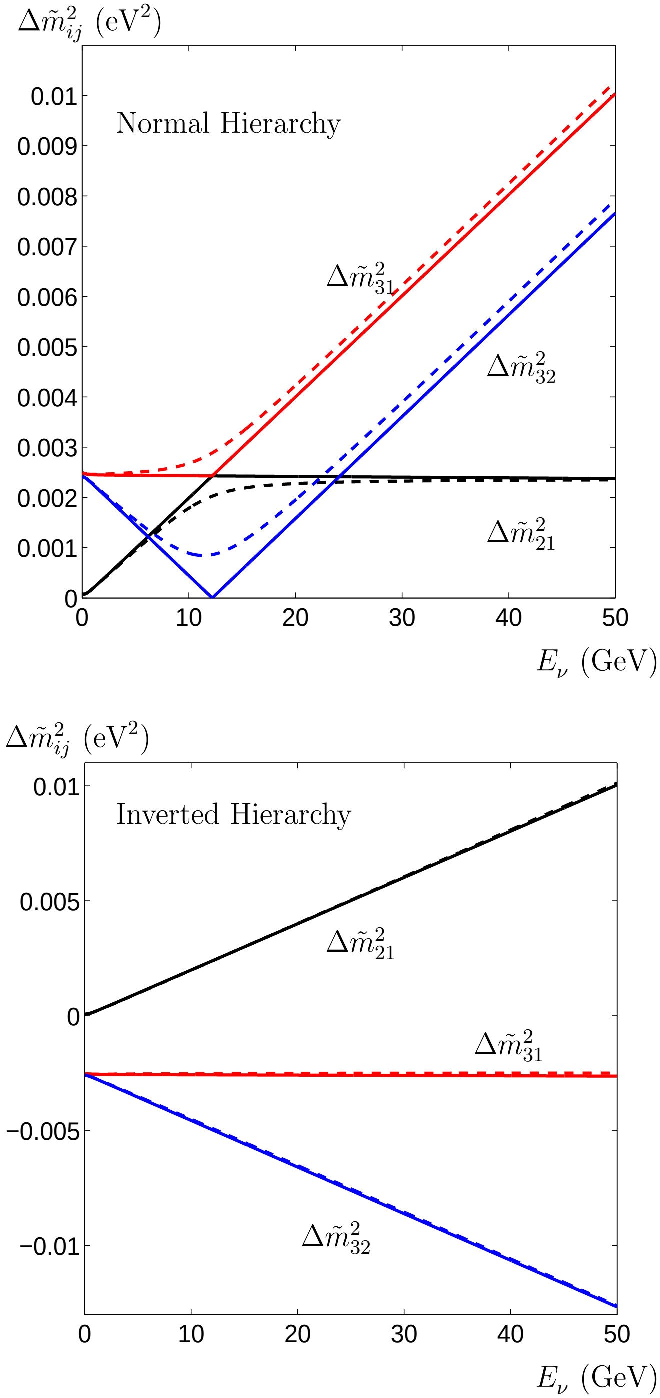
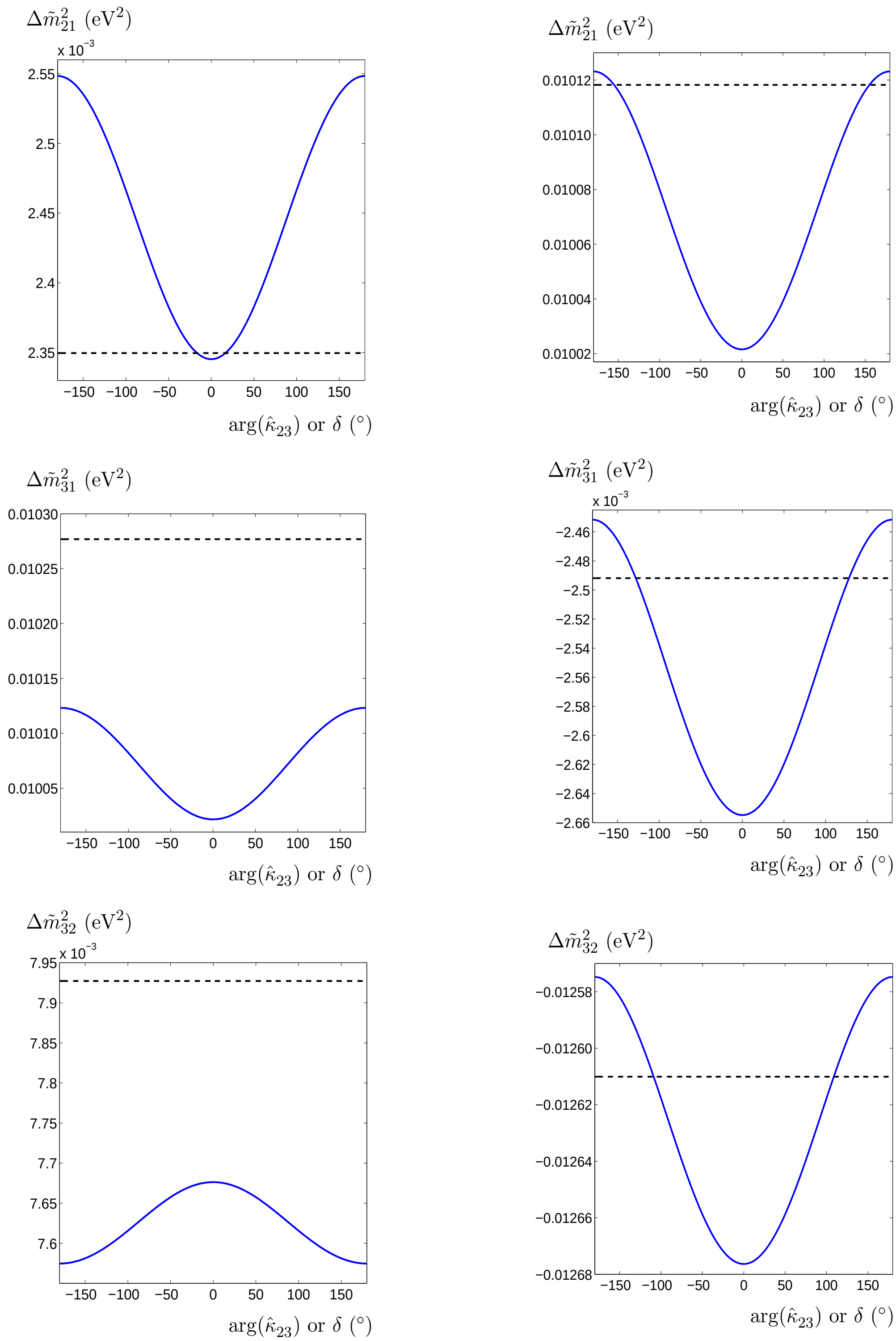


FIG. 1: The effective mass-squared differences in matter as functions of the neutrino beam energy E_ν in Case I (the unitary case, represented by dashed lines) and Case II (the non-unitary case, represented by solid lines) for both the normal (the first plot) and the inverted (the second plot) hierarchies.



(a) Normal hierarchy

(b) Inverted hierarchy

FIG. 2: The effective mass-squared differences in matter as functions of the Dirac phase δ in Case I (the unitary case, represented by dashed lines) or the phase of $\hat{\kappa}_{23}$ in Case II (the non-unitary case, represented by solid lines) for both mass hierarchies, where we choose $E_\nu = 50$ GeV.

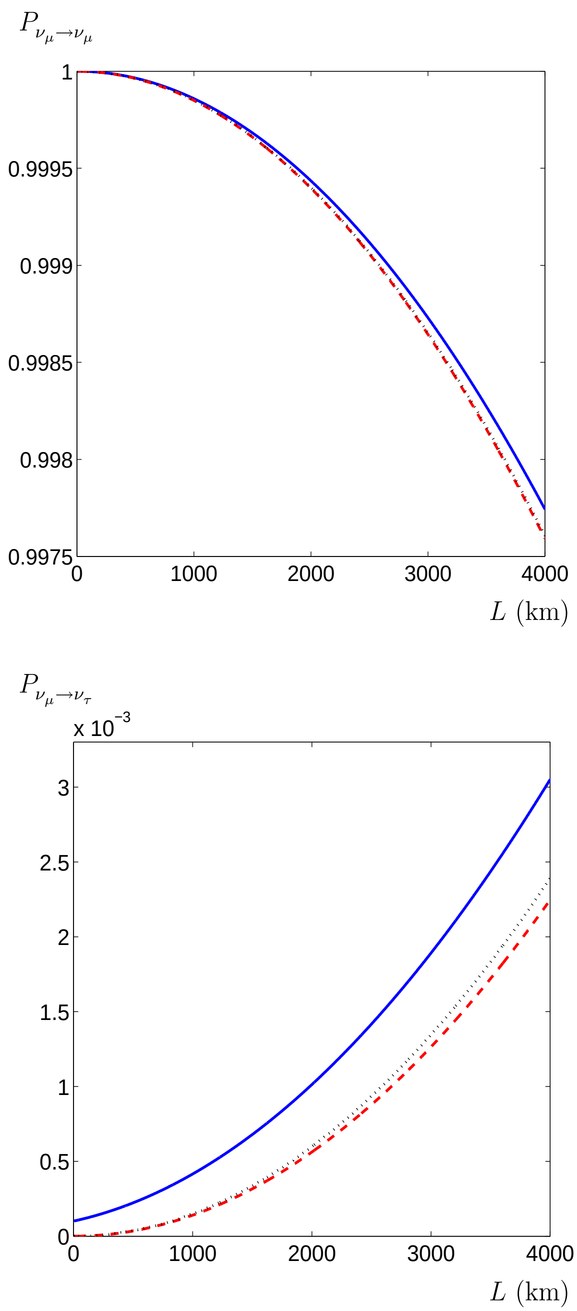


FIG. 3: The probabilities of neutrino oscillation $\nu_\mu \to \nu_\mu$ (the first plot) and $\nu_\mu \to \nu_\tau$ (the second plot) in matter as functions of the baseline L in Case I (the unitary case, represented by dashed lines) and Case II (the non-unitary case, represented by solid lines), in the normal hierarchy case. Here we choose $E_\nu = 50$ GeV. The dotted lines in the figure show the corresponding probabilities if the neutrino mixing is the exact tri-bimaximal mixing.

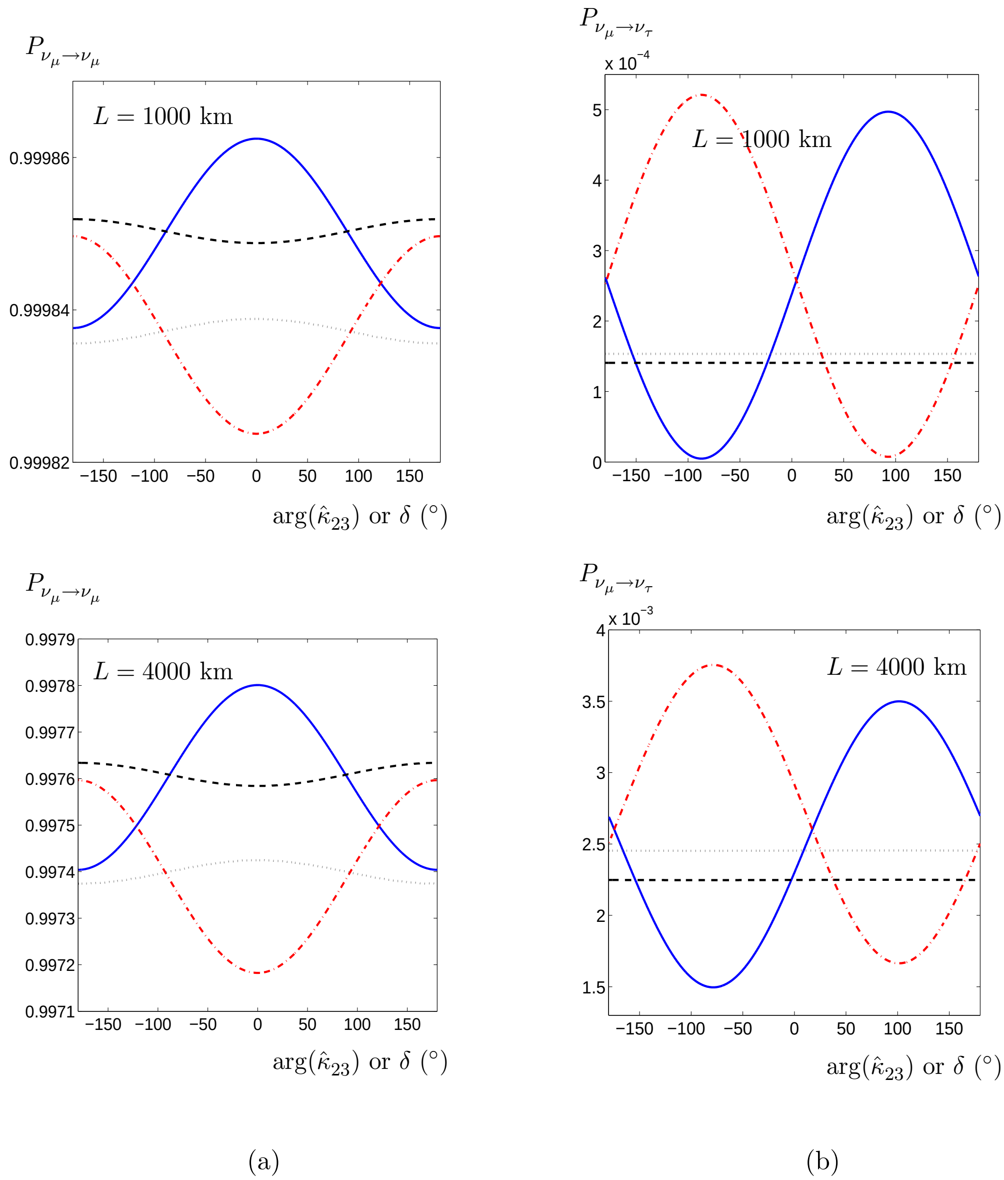


FIG. 4: The probabilities of neutrino oscillation $\nu_\mu \to \nu_\mu$ (a) and $\nu_\mu \to \nu_\tau$ (b) in matter as functions of the Dirac phase δ in Case I (the unitary case, the dashed lines for the normal hierarchy, the dotted lines for the inverted hierarchy) or the phase of $\hat\kappa_{23}$ in Case II (the non-unitary case, the solid lines for the normal hierarchy, the dot-and-dash line for the inverted hierarchy), where we choose $E_\nu = 50$ GeV.